Cooperative games

\[
\begin{array}{cc}
(0,0) & (0,0) \\
(0,0) & (3,0)
\end{array}
\]

Players choose actions \( i, j \) in agreement

- Declare that if agreement is broken,
  - Ruth uses \( x \)
  - Chris uses \( y \)

Transferable Utility: agreement can include a side payment. (TU)

Non Trans. Utility: No side payments possible (NTU)

Credible threat: person making the threat is hurt less than opponent.
Transferable Utility

E.g. 

\[
\begin{bmatrix}
4 & 5 \\
3 & 2
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
5 & 1
\end{bmatrix}
\]

If \((x,y)\) feasible, so is \((x+a,y-a)\) for \(a \in \mathbb{R}\).

Pareto-optimal outcome:

Outcome \((u,v)\) so that no other feasible outcome \((u',v')\) has \((u' \geq u, v' \geq v)\).

Set of outcomes

Empty

Pareto optimal.
How do they split $4 + 5 = 9$?

E.g. If Ruth's threat is $(0,1)$ i.e. row 2 (R)
Chris's threat is $(\frac{5}{2}, \frac{1}{2})$ (Y)

Ruth gets $x^T A y = \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 5 = 4$
Chris gets $x^T B y = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 1 = \frac{3}{2}$

Pareto optimal, $U + V = 9$

Ruth's threat is credible
Chris's threat is credible

$(4, \frac{3}{2})$ is the threat.

Outcome will be $(U, V)$ where $U + V = 9$

$U - 4 = V - \frac{3}{2}$
$U - V = 4 - \frac{3}{2}$

$U - 4 = (9 - U) - \frac{3}{2}$
$2U = 13 - \frac{3}{2}$

$U = \frac{23}{4} \quad V = \frac{13}{4}$
If threats are $x,y$, resulting outcome is

$$(x^t Ay, x^t By)$$

Agreement will have $U + V = M \leftarrow \max A_{ij} + B_{ij}$.

$U - x^t Ay = V - x^t By$, otherwise one of players will demand more, and have credible threat.

$$U = \frac{M + (x^t Ay - x^t By)}{2} \quad V = \frac{M - (x^t Ay - x^t By)}{2}$$

Let $C = A + B$. Ruth wants to maximize $x^t Cy$.

Chris wants to minimize $x^t Cy$.

This is a zero-sum game with matrix $C$. 
In game $\begin{pmatrix} 45 & 60 \\ 32 & 51 \end{pmatrix}$, $C = A - B = \begin{pmatrix} -1 & 0 \\ 1 & 4 \end{pmatrix}$

Here, NE is (row 2, col 1) (in 0-sum game C). Optimal threats are row 2, col 1.

{threat}\{point\} is (3, 2), $U - V = 3 - 2$.

$U + V = 9$

$U = 5$, $V = 4$

e.g. $\begin{pmatrix} 4,6 \\ 7,2 \end{pmatrix}$, $\begin{pmatrix} 4,1 \\ 0,0 \end{pmatrix}$, $\begin{pmatrix} 5,5 \\ 0,4 \end{pmatrix}$

Pareto optimal results have $U + V = 10$

$C = A - B = \begin{pmatrix} -2 & 3 & 0 \\ 5 & 0 & -4 \end{pmatrix}$

After domination, $\begin{pmatrix} -2 & 0 \\ 5 & -4 \end{pmatrix}$

$-2x_1 + 5x_2 = -4x_2$

$2x_1 = 9x_2$

$x = (\frac{9}{11}, \frac{2}{11})$

Value is $-8/11 = x^TCy$
The agreed outcome will be:

\[
\frac{10 + \left(-\frac{8}{11}\right)}{2} = \frac{57}{11}
\]

\[
\begin{align*}
U+V &= 10 \\
U-V &= -\frac{8}{11}
\end{align*}
\]

\[
V = \frac{10 - \left(-\frac{8}{11}\right)}{2} = \frac{59}{11}
\]

Threats:
- Ruth will use \( \left(\frac{9}{11}, \frac{2}{11}\right) \)
- Chris will use \( \left(\frac{4}{11}, 0, \frac{7}{11}\right) \)

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Note: T.V. case

Assume there is a given set of feasible outcomes.

Assume a disagreement point \( (U_0, V_0) \)
Problem 3 sol.

\[
\begin{array}{ccc}
(2,3) & (0,1) & (2,0) \\
(0,3) & (3,2) & (0,0) \\
(2,2) & (3,0) & (2,3) \\
\end{array}
\]

find all NE.

row 3 dominates (weakly) 1, 2

col 2 strictly dominated by col. 1.

so eliminate col. 2.

Now row 3 strict. dom. row 2

This leaves

\[
\begin{array}{ccc}
1 & (2,3) & (2,0) \\
3 & (2,2) & (2,3) \\
\end{array}
\]

\[3p + 2(1-p) = 2 + p\]

Pure NE (1,1) (3,3)

If Ruth uses \((p, 0, 1-p)\), Chris has payoff vector \((2+p, \underline{20+p}, 3(1-p))\)

If \(p < \frac{1}{4}\), Chris uses 1

\[p \leq \frac{1}{4}\]

\[p = \frac{1}{4}\]

\[p = \frac{1}{4}\] any \((9, 0, 1-9)\)