Today: Non Transferrable Utility in cooperative games.

TV: how to split largest total possible outcome

Feasible set in TV

Threat point \((u_0, v_0)\) has

\[ u_0 - v_0 = \text{Val}(A-B) \]

NTV Feasible set \(S\)

Assume: \(S\) convex, bounded.

\((u_0, v_0)\): disagreement point.
\[ u^*, v^* = \left( \frac{\sqrt{2}}{z}, \frac{\sqrt{z}}{z} \right) \]

\[ u, v \rightarrow 3u, v+1 \]

\[ \left( \frac{3\sqrt{2}}{z}, \frac{\sqrt{z}}{z}+1 \right) \]

\[ (0, 0) \]

\[ (1, 1) \]
First step: For given \((u_0, v_0)\) disagreement, what is the solution of the game? 
\[ Y(S, u_0, v_0) \]

Second step: Given \( Y \) each player tries to make a threat \( a \) to maximize their result.

Nash Bargaining

Let \((u^*, v^*) = Y(S, u_0, v_0)\)

Axioms:
1. Feasible: \((u^*, v^*) \in S \cap \{u^* \geq u_0, v^* \geq v_0\}\)
2. Pareto-Optimal: no other \((u, v)\) in \( S \) has \( u \geq u^* \) \& \( v \geq v^* \)
3. Symmetry: If game is symmetric to reflection on the line \( u = v \), then \( u^* = v^* \)
4. Invariance to affine maps: translation + scaling.

If \( \hat{S} = S + (a, b) \)
\[ Y(\hat{S}, u_0 + a, v_0 + b) = (u^* + a, v^* + b) \]

If \( \hat{S} = \{a x, b v\} \circ (u, v) \in S \)
\[ Y(\hat{S}, au_0, bv_0) = au^*, bv^* , \ a, b > 0. \]
5. IIA: Independence of Irrelevant Alternatives.

If \( T \subseteq S \) and \( (\mathbf{u}^*, \mathbf{v}^*) = y(S, u_0, v_0) \) and \( (\mathbf{u}^*, \mathbf{v}^*) \in T \) then \( (\mathbf{u}^*, \mathbf{v}^*) = y(T, u_0, v_0) \).

\[ \begin{align*}
S & \quad (\mathbf{u}^*, \mathbf{v}^*) \\
\mathbf{u}_0 \quad \mathbf{v}_0 & \quad T
\end{align*} \]

**Thm** The only \( y \) satisfying the axioms is \( (\mathbf{u}^*, \mathbf{v}^*) \) maximizes \( (\mathbf{u}^*-\mathbf{u}_0)(\mathbf{v}^*-\mathbf{v}_0) \).

This is the Nash bargaining solution.
\[ \text{e.g. } V_0 = 0, S = \text{conv}(0,0,1,8,6,4,7,9) \]

\[ (6, 4) \]

\[ (0, 0) \]

\[ (7, 0) \]

On top side, \( V = 8 - \frac{2}{3}u \), 

\[ (V-V_0)(V-V_0) = V(8 - \frac{2}{3}u) \] maximized when

\[ 8 - \frac{4}{3}u = 0 \]

\[ u = 6 \]

On right side, \( V = -4u + 28 \)

\[ u \cdot V = -4u^2 + 28u \]

\[ \frac{du}{dV} = -8u + 28 \]

\[ \frac{du}{dV} < 0 \] for \( u \in [6, 7] \) so max at \( u = 6 \)

If disagreement pt is \((6, 1)\),

On top, \( (U-1)(V-1) = (U-1)(8 - \frac{2}{3}U - 1) = -\frac{2}{3}U^2 + \frac{23}{3}U + 7 \)

\[ \frac{du}{dV} = -\frac{4}{3}U + \frac{23}{3} \]

crit. pt. is \( u = \frac{23}{4} \)
\[(0^*, W^*) = \left( \frac{23}{4}, \_ \right) \]

**Proof of Thm.** 1. Assume \( U_0 = V_0 = 0 \), otherwise translate.

2. There is a unique point in \( S \) that maximizes \( U \cdot V \).
If two points \((u, v) (u', v') \in S\) have same \(uv = u'v'\), then points on the segment have higher values: \(u'' = \frac{u + u'}{2}, \ v'' = \frac{v + v'}{2}\)

\(u'' v''\) is above hyperbola \(uv = \text{const}^2\)

3. Assume \((u^*, v^*) = (1, 1)\) otherwise rescale.
In triangle $\mathcal{T} = \{(u, v) : 0 \leq u, v, u + v \leq 2\}$ by symmetry $(u^*, v^*) = (1, 1)$.

4. If $S \subset T$ and $(1, 1) \in S$ by II A $\mathcal{P}(S, 0, 0) = (1, 1)$.

5. If after scaling $(1, 1)$ maximises $u \cdot v$ in $S$ then $S \subset T$.

$$\forall u, v \in S, u \cdot v \leq 1$$

If $z \in S$ not in $T$.

The segment $z - (1, 1) \subset S$. This segment is not below $\{u \cdot v = 1\}$. \qed