Problem 1. Find the optimal agreement and a corresponding threat strategies of the following cooperative T.U. games:

\[
\begin{array}{cccc}
(1,2) & (0,1) & (8,0) & (3,6) \\
(5,5) & (3,0) & (1,1) & (0,3) \\
(1,1) & (0,4) & (2,1) & (1,3) \\
(2,0) & (4,2) & (3,0) & (5,1)
\end{array}
\quad
\begin{array}{cccc}
(5,2) & (2,1) & (1,1) & (0,0) \\
(0,0) & (0,5) & (2,0) & (3,1) \\
(3,0) & (1,5) & (1,5) & (2,2)
\end{array}
\]

**Solution:** (a) The maximal total utility is 5 + 5 = 10, so the agreement will have \( u + v = 10 \). The threats are optimal strategies in the game

\[
\begin{pmatrix}
-1 & -1 & 8 & -3 \\
0 & 3 & 0 & -3 \\
0 & -4 & 1 & -2 \\
2 & 2 & 3 & 4
\end{pmatrix}
\]

Row 4, column 1 is a saddle point with value 2, so \( u - v = 2 \), and hence \( (u, v) = (6, 4) \), and the threats are row 4, column 1. Alternatively: Row 3 is dominated. Then column 3 is dominated. Then row 1, column 2, row 1 and column 4, leaving a 1 \times 1 \) game with outcome 2.

(b) Here we have \( u + v = 5 + 2 = 7 \).

\[
\begin{pmatrix}
3 & 1 & 0 & 0 \\
0 & -5 & 2 & 2 \\
3 & -4 & -4 & 0
\end{pmatrix}
\]

Row 3 is dominated, and then column 1,4 are dominated, leaving the game \( \begin{pmatrix} 1 & 0 \\ -5 & 2 \end{pmatrix} \), with value 1/4 and threat strategies \( \left( \frac{7}{8}, \frac{1}{8}, 0 \right) \) and \( \left( 0, \frac{1}{4}, \frac{3}{4}, 0 \right) \) (Latter is not unique.) The outcome has \( u - v = \frac{1}{4} \) so \( u = \frac{29}{8} \) and \( v = \frac{27}{8} \).

Problem 2. Suppose the feasible outcome set of a cooperative NTU game is the quadrangle with corners \( (0, 0), (7, 0), (6, 1), (0, 4) \).

(a) Find the Nash bargaining solution if the disagreement point is \( (0, 0) \).

(b) Find the Nash bargaining solution if the disagreement point is \( (3, 0) \).

**Solution:** (a) The maximum of \( uv \) along the segment \( (0, 4) - (6, 1) \) is at \( (4, 2) \). The maximum along \( (6, 1) - (7, 0) \) is at \( (6, 1) \) so the maximal value at a pareto-optimal point is at \( u^a, v^* = (4, 2) \).

(b) The maximum of \( (u - 3)v \) along the boundary is at \( (5.5, 1.75) \).
**Problem 3.** Find the Nash solution of a cooperative NTU game with feasible outcome set \{ (u, v) : u^2 + v^4 \leq 1 \} with disagreement point \((-1, 0)\).

**Solution:** We wish to maximize \((u + 1)v\) along the curve \(u^2 + v^4 = 1\), between the points \((0, 1)\) and \((1, 0)\). Writing \((u + 1)v = (u + 1)(1 - u^2)^{1/4}\) and equating the derivative to 0 leads to \(3u^2 + u - 2 = 0\), so \(u = \frac{2}{3}\), \(v = (\frac{5}{9})^{1/4}\).

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**Problem 4.** We consider the NTU cooperative game

\[
A, B = \begin{pmatrix}
(4,0) & (2,4) \\
(0,6) & (0,0)
\end{pmatrix}
\]

(a) Draw the set of feasible outcomes.
(b) Partition this into three sets \(A, B, C\), so that if the disagreement point is \(A\) the Nash bargaining outcome is the point \((2, 4)\), if the point is in \(B\) it is on the segment \((2, 4) - (4, 0)\) and if in \(C\) it is on the segment \((2, 4) - (0, 6)\). (c) Show that player 1 has a threat strategy that forces the outcome to be in \((2, 4) - (4, 0)\)
(d) Show that player 2 has a threat strategy that forces the outcome to be \((2, 4)\).
Conclusion: the outcome of this NTU game will be \((2, 4)\).
(c) What is the outcome as a TU cooperative game?

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**Solution: (a,b) See above.**

(c) If player 1 picks row 1, the outcome is along the segment \((4,0) - (2,4)\) (depending on player 2’s strategy). In all cases the NTU outcome is the same point.
(d) Using column 2, every strategy of player 1 gives Nash bargaining outcome (2, 4).
(e) Here $u + v = 6$ and $u - v = -1$ so $u = 2.5$ and $v = 3.5$. 