Problem 1. In a gameshow, Ruth has $200 and Chris has $300. Each must decide to pass or gamble, not knowing the choice of the other. A player who passes keeps the money they started with. If Ruth gambles, she wins $200 with probability 1/2 and loses $100 with probability 1/2. (a) Describe this in matrix form. (b) Draw the Kuhn (game) tree, indicating information sets. (c) Find the safety levels. (d) Find a Nash equilibrium.

solution (a) If both gamble, Ruth has either $100 or $400 and Chris either $0 or $600. Chris will get the bonus if his gamble wins and Ruth if not, so on average each gets $200 of the bonus. Therefore the average outcomes in this case are $(450,500)$. The other cases are similar and give the following values:

<table>
<thead>
<tr>
<th></th>
<th>Y</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>$(450,500)$</td>
<td>$(450,500)$</td>
</tr>
<tr>
<td>N</td>
<td>$(400,500)$</td>
<td>$(200,700)$</td>
</tr>
</tbody>
</table>

(b) The tree:

(c) Ruth gets $450 by gambling and less by not gambling, so safety level is $450. Chris’s safety level is $500. (d) Gambling is strictly dominating for Ruth, so she will gamble in any Nash equilibrium. Any strategy for Chris will be a N.E. with this.

Problem 2. Find a Nash equilibrium in the game

\[
\begin{pmatrix}
(1, 2) & (0, 1) & (0, 1) \\
(3, 0) & (1, 1) & (0, 3) \\
(2, 0) & (3, 0) & (5, 1)
\end{pmatrix}
\]

Solution: Deterministically picking row 3 and column 3 is the unique N.E. To see this, note that row 1 is strictly dominated by row 3, so is never used. After removing it, columns 1,2 are strictly dominated by column 3, so only column 3 can be used, and therefore only row 3.
Problem 3. Find all Nash equilibria in the game

\[
\begin{array}{ccc}
(2,3) & (0,1) & (2,0) \\
(0,3) & (3,2) & (0,0) \\
(2,2) & (3,0) & (2,3)
\end{array}
\]

Solution: Column 2 is strictly dominated by column 1, so is not used in any N.E.. Since Chris does not use column 2, row 2 is strictly dominated by the others, and is not used in the N.E. This leaves rows 1,3 and columns 1,3. Ruth always gets 2, so it does not matter to her what she does. Suppose she uses strategy \((p, 0, 1-p)\). Then Chris has payoff vector \((2 + p, 3 - 3p)\). If \(p > 1/4\) he must use column 1. If \(p < 1/4\) he uses column 3. If \(p = 1/4\) he can use any strategy \((q, 0, 1-q)\).

Note: row 3 dominates the other two rows, but only weakly. If Ruth uses row 2 in a N.E. then Chris must use only column 2, but that would not be his optimal response. If Ruth uses rows 1 then Chris uses columns 1,3 and we find the previous equilibria.

Problem 4. Solution: (a) The conditions for \((x,y)\) to be a N.E. are that \(y\) is an optimal response to \(x\), and that \(x\) is an optimal response to \(y\). With a fixed \(x \in \Delta^m\), let \(T_x\) be the set of actions for player 2 which give the maximal result opposite \(x\), i.e. the set of maximal entries of \(x^T B\). The first condition is simply that \(y_i = 0\) for \(i \notin T_x\), which is a convex set of \(y\)'s. Let \(A_x\) be the set of actions of player 1 which have a positive probability in \(x\). The second condition requires that any \(i \in A_x\) gives player 1 at least as much as any other action, that is for any \(i \in A_x\) and any \(j\) we have

\[
\delta_i^T Ay \geq \delta_j^T Ay.
\]

This is a linear inequality on the entries of \(y\), and there is one for each \(i,j\). Therefore the set \(S\) is the set of \(y\)'s which satisfy 0 outside of \(T_x\) and satisfy a number of linear inequalities. Such a set is convex, since a convex combination of solutions is also a solution.

(b) Almost any game with multiple N.E. is an example. Consider the battle of the sexes game. The N.E. are \((1,1), (2,2)\), and a unique mixed N.E. \(x = (1/3, 2/3)\) and \(y = (2/3, 1/3)\). This is not convex.

Problem 5. We consider a model for a duopoly in a new product. If the total amount produced is \(Q\), then the price of each unit is \(A - Q\), for some fixed and known \(A\). (a) Company I pays \(C_1\) to make each unit. How much should they produce (to maximize profit)?

(b) Company II enters the market, and can produce each unit for only \(C_2 < C_1\). Suppose company I decides how much to produce and declares the decision. Then company II decides how much to produce. Find all Nash equilibria for this model, and compare the profits and price.
Solution: (a) The profit of company I is $Q(A - Q - C_1)$, maximized when $Q = \frac{A - C_1}{2}$ if this is positive with value $\frac{(A - C_1)^2}{4}$. If $C_1 \geq A$ then company I will not produce anything even with no competition (obviously).

(b) If the companies produce $Q_1, Q_2$, then given $Q_1$, company II’s profit is $Q_2(A - Q_1 - Q_2 - C_2)$, maximized at $Q_2 = \frac{A - Q_1 - C_2}{2}$ (assuming this is positive). Note that company I will always produce $Q_1 \leq A - C_1 < \bar{A} - C_2$, so this $Q_2$ is indeed positive.

The profit of company I is therefore

$$Q_1(A - Q_1 - Q_2 - C_1) = Q_1(A - Q_1 - \frac{A - Q_1 - C_2}{2} - C_1) = \frac{Q_1(A - Q_1 - 2C_1 + C_2)}{2}.$$ 

This is maximized at $Q_1 = \frac{A - 2C_1 + C_2}{2}$, assuming this is positive.

Summary: If $A > 2C_1 - C_2$ then company I produces $\frac{A - 2C_1 + C_2}{2}$. If $A \leq 2C_1 - C_2$ then company I should not produce anything.