Problem 1.  (a) Find the safety levels and all Nash equilibria for the game

\[
\begin{array}{cc}
(1,1) & (2,4) \\
(4,4) & (1,0)
\end{array}
\]

(b) Find the unique mixed Nash equilibrium for the following game.

\[
\begin{array}{cc}
(3,-3) & (0,4) \\
(2,1) & (1,0)
\end{array}
\]

(c) Find the safety levels and all pure Nash equilibria for the following game.

\[
\begin{array}{ccc}
(3,1) & (0,3) & (4,5) \\
(1,5) & (3,1) & (1,3) \\
(4,3) & (4,5) & (1,1)
\end{array}
\]

(Bonus d) Is there a mixed Nash equilibrium? Find one if so.

Solution: (a) Ruth’s safety level is $7/4$, with strategy $(3/4, 1/4)$. Chris’s safety level is $16/7$ with strategy $(4/7, 3/7)$.

There are pure N.E. $(1,2)$ and $(2,1)$ (with outcomes $(4,4)$ and $(2,4)$). The only mixed N.E. is $(4/7, 3/7)$ for Ruth and $(1/4, 3/4)$ for Chris.

(b) If Chris uses $(1/2, 1/2)$ then Ruth is indifferent to the two rows. If Ruth uses $(1/8, 7/8)$ then Chris is indifferent, so that is the N.E.

(c) Ruth can ignore row 2 (dominated by 3) and has safety level $16/7$ with strategy $(3/7, 0, 4/7)$. Chris has safety level 3 with strategy $(1/3, 1/3, 1/3)$. The two entries $(4,5)$ are pure N.E.

(d) If there is a mixed N.E., we can assume it does not use the dominated row 2. Since it must use both other rows, Ruth is indifferent between them. If Ruth does not use row 2, then for Chris column 2 dominates column 1, so he will not use column 1. If Chris uses $(0, p, 1-p)$, then Indifference for Ruth gives $4(1-p) = 4p + (1-p)$, so Chris uses $(0, 3/7, 4/7)$. Similarly, Ruth uses $(2/3, 0, 1/3)$.

Problem 2: volunteer dilemma  This is an $n$ player game. Each player decides whether to volunteer or not, all at the same time. If no player volunteers they all get 0. If some volunteer, the volunteers get 1 and the non-volunteers get 2.

(a) Show that the strategies where one player always volunteers and the rest never do is a Nash equilibrium.

(b) Show that there is a (unique) symmetric Nash equilibrium, where every player volunteers with the same probability $p$, and find the expected payoff in this equilibrium.
Solution: (a) If all others do not volunteer, you are better off volunteering. If at least one
other player volunteers, you are better off not volunteering, so that is a N.E.
(b) If every player volunteers with probability \( p \), the probability that none else volunteers
is \( (1 - p)^{n-1} \). If you volunteer you get 1 and if not you get \( 2(1 - (1 - p)^{n-1}) \). These are
equal if and only if \( (1 - p)^{n-1} = 1/2 \), so \( p = 1 - (1/2)^{1/(n-1)} \). The expected payoff is 1.

Problem 3. Three players pick a number from \( \{1, 2, \ldots, 10\} \). The winner is the player who picked
the median, and they get $1. If more than one person picked the median, all who picked the median
win. Find a Nash equilibrium for this game.

Solution: All players picking the same number \( i \) is a N.E. (for any \( i \)).

There are no other pure N.E.. However, there are some mixed N.E.: for example everyone
picks either 1 or 2 randomly.

Problem 4. Two companies make bids for a project, and the lower bid gets hired. In case of tie,
one is chosen at random. The payoff for a company not hired is 0. The payoff for Company A if
hired is \( B_A - 10 \), and for company B it is \( B_B - 12 \), where \( B_A, B_B \) are the bids made (and 10,12
are the costs to build the project. Suppose the bids can be any positive integer. Find all Nash
equilibria for this game.

Solution: Each company can be certain to get 0, (A by bidding 10 and B by bidding 12), so
no lower bid is ever accepted. If company B bids \( s \) then the optimal reply for A is

\[
\begin{aligned}
&\text{if } s - 1 \geq 13, \\
&\text{if } 11 \leq s < 12, \\
&\text{if } 11 = s = 12, \\
&\text{if } \text{any bid } \geq 10 \text{ and } s = 10, \\
&\text{if } \text{any bid } \geq s > 10.
\end{aligned}
\]

Note that company B should not bid less than 12, but might. If A bids \( s \) then the optimal
reply by B is

\[
\begin{aligned}
&\text{if } s - 1 \geq 15, \\
&\text{if } 13 \leq s < 14, \\
&\text{if } 13 = s = 13, \\
&\text{if } \text{any bid } \geq 12 \text{ and } s = 12, \\
&\text{if } \text{any bid } \geq s > s < 12.
\end{aligned}
\]

From this we see that the pairs of mutually optimal replies are \( (11, 12), (12, 12), (12, 13) \).

We can find some mixed N.E. easily: If company A bids either 11 or 12, with any probabil-
ities then company B can do no better than bidding 12, and this is a N.E.. Similarly, there is
a N.E. where A bids 12 and B bids 12 or 13 with any probabilities.
There are many other N.E., where company A bids 12, and company B makes some higher bids. For example, if B bids 13 with probability at least 2/3 and otherwise 14, this is a N.E. We can characterize all N.E. where A bids 12 and B makes a random bid using a system of inequalities, but the full set of solutions is hard to describe.

Try out http://cgi.csc.liv.ac.uk/cgi-bin/cgiwrap/rahul/input.py for a general solver.

Problem 5. A constant-sum game is a game where there is some $L$ so that $a_{ij} + b_{ij} = L$ for every $i, j$. Prove that in a constant-sum game every Nash equilibrium gives Ruth the same payoff (and similarly for Chris).

Solution: Consider the game $A, B'$ where $B'_{ij} = B_{ij} - L$. This is the same as playing the given game, and then Chris pays a fixed amount $L$. Every N.E. in the game is also an equilibrium in the new game, since the fixed payment does not change any preferences. The new game is a 0 sum game, so every N.E. has the same value $V$, and therefore the same holds for the original game. (The payoffs are $V$ and $L - V$.)