0 - sum games

Strategy: A rule for making all decisions.

e.g. if 

\[
\begin{align*}
& \text{"Always take 1 chip" } \\
& \begin{cases} 
& \text{If } n \equiv 0 \mod 3 \quad \text{take 1} \\
& \quad = 1 \mod 3 \quad \not\rightarrow 1 \\
& \quad = 2 \mod 3 \quad \not\rightarrow 2 
\end{cases} \\
& \text{copy opponent: repeat their last move.} \\
& \text{on turn 1 take 1.} \\
& \text{toss a coin. } H \rightarrow 1 \text{ chips.} \\
& \quad \text{T } \rightarrow 2 \\
\end{align*}
\]

many others: too many to list.
Given the two strategies outcome is non-random
determined.

\[
\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & 5 & \cdots \\
\hline
1 & -1 & 0 & 1 & 0 & 1 & \cdots \\
2 & 0 & 1 & 0 & 1 & 0 & \cdots \\
3 & 1 & 0 & 1 & 0 & 1 & \cdots \\
4 & 0 & 1 & 0 & 1 & 0 & \cdots \\
\end{array}
\]

\[
A_{ij} = \begin{cases} 
1 & \text{if player 1 wins} \\
-1 & \text{if player 2 wins} \\
0 & \text{if draw}
\end{cases} \\
\text{when player 1 uses strategy } i \text{ and player 2 uses strategy } j.
\]
A zero-sum game is given by an array \( A \). A \( n \times m \) array if player 1 has \( n \) strategies and player 2 has \( m \) strategies. Each player chooses a strategy simultaneously, and the outcome is \( A_{ij} \).

Zero-sum: gain of 1 is loss of 2. Player 2 pays \( A_{ij} \) to player 1.

In a combinatorial game, \( A_{ij} \in \{1, -1, 0\} \).

→ A winning strategy for player 1 is some \( i \) such that \( A_{ij} = 1 \) for all \( j \).

→ A winning strategy for player 2 is a column of \(-1\)'s. \( \forall j \) such that

\[ \forall i \ A_{ij} = -1 \]
Let $A$ be an $n \times m$ matrix.

Play that game.

\[ e.g. \ A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \]

\[
\begin{array}{c|ccc}
 & R & P & S \\
\hline
R & 0 & -1 & 1 \\
P & 1 & 0 & -1 \\
S & -1 & 1 & 0 \\
\end{array}
\]

\[ (1, 0) \]

\[ (0, 1) \]

\[ e.g., \ \text{Loomie/twowine game:} \]

\[
\begin{pmatrix} L & T \\ L & (1 \ 0) \\ T & (0 \ 2) \end{pmatrix}
\]
Player 1 can guarantee $\geq 0$
Player 2 can guarantee $\leq 1$

Same bounds for $\begin{pmatrix} 1 & 0 \\ -5 & 2 \end{pmatrix}$

Bound for Player 1:

If choose $i$, worst case is

$$\min_{j} \ A_{ij}$$

So Player 1 can always achieve at least

$$\max \ \min_{i,j} \ A_{ij}$$

\[
\begin{pmatrix}
5 & 0 & -3 & 2 & 8000 \\
-6 & -2 & 6 & 4 & 1 \\
-8 & 1 & 0 & 5 & 0 \\
1000 & 0 & 2 & -1 & 6 \\
\end{pmatrix}
\]

$$\min_{i,j} A_{ij}$$

$-3$

$-6$

$-8$

$-1$
Player 2 can guarantee \[
\min \max_{j} \max_{i} A_{ij}\]

**Theorem** In any $A$,
\[
\max \min_{i} \min_{j} A_{ij} \leq \min \max_{j} \max_{i} A_{ij}
\]

Outcome if player 1 reveals move first.

Proof: Let $i^*$ be maximizer of $\min_{j} A_{i^*j}$

Let $j^*$ be minimizer of $\max_{j} A_{ij^*}$

Consider $A_{i^*j^*}$

$\max \min = \min_{i} \max_{j} A_{i^*j^*} \leq A_{i^*j^*} \leq \max_{i} \min_{j} A_{ij^*} = \min \max_{i} A_{ij^*}$
Sometimes, $\max \min = \min \max$. Called a saddle point.

**Def:** A saddle point is $(i, j)$ which has:

- $A_{ij}$ is smallest in row $i$.
- $A_{ij}$ is largest in column $j$.

$$
\begin{pmatrix}
*, *, 1, *, * \\
*, *, 2, *, * \\
4, 6, 3, 10, 3 \\
*, *, -1, *, * \\
*, *, 0, *, *
\end{pmatrix}
$$

$\min_j A_{ij}$

If $(i, j)$ is a saddle point then $i, j$ are optimal choices.