Plan:

- Proof of Brouwer fixed pt. theorem
  Using Sperner's Lemma
- Monotonicity of games
- Commitments
- Partial information
- Midterm comments
Recall Sperner’s Lemma

Corners are
• each side
has only 2 colours
Inside: Anything goes

Lemma: \exists odd number of R-Blue-Black
triangles. In particular, not 0.
Brouwer's fixed pt theorem:
If K closed, convex, bounded, and f: K → K continuous, then f has a fixed point x such that f(x) = x.

Proof. Case K = \( \Delta^2 \) = \( \{(xy, z) : x + y + z = 1, x, y, z \geq 0\} \)

\[ f(x, y, z) = (\alpha, \nu, w) \]

Partition \( \Delta \) into small \( \Delta \)'s of size \( \leq \varepsilon \)

\((x, y, z)\) is Red if \( \nu > x \)

Blue if \( \nu < x \) and \( \nu < y \)

Black if \( \nu < x \) and \( \nu \geq y \) and \( w < z \)
If \((u,v,w) = (x,y,z)\) we have a fixed point. Either \(u < x\) or \(v < y\) or \(w < z\) so every point gets a colour.

Sperner: \(\exists\) rainbow triangle

At \((1,0,0)\), \(u < 1\)

At \((p, 1-p, 0)\) one of the first two coord. must decrease
Find points $a_\epsilon$, $b_\epsilon$, $c_\epsilon$ s.t.

$f(a_\epsilon)$ has 1st coord. $< f$st coord. of $a_\epsilon$

$f(b_\epsilon)$ 2nd " $< 2$nd " $b_\epsilon$

$f(c_\epsilon)$ 3rd " $< 3$rd " $c_\epsilon$

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Prove this for $\epsilon = \frac{1}{n}$ $n=1,2,3$...

Find sub sequence $a_{n_k} \to a$,

$b_{n_k} \to b$

$c_{n_k} \to c$
Claim: \( a \) is the fixed point.

Let \( f(a) = (u, v, w) \) \( \quad a = (x, y, z) \)

\[
\begin{align*}
\text{for } \mathbf{a} & \quad u \leq x \\
\text{for } \mathbf{b} & \quad v \leq y \\
\text{for } \mathbf{c} & \quad w \leq z
\end{align*}
\]

but \( u + v + w = 1 = x + y + z \) so \( (x, y, z) = (u, v, w) \) so we have a f.p. \( \square \)

\[ f: O \rightarrow \]

\[ f(x) = f(y) \]
Hairy Ball theorem

At every $x$ in 2-dim sphere, pick a direction tangent to sphere.

Thm: There must be a discontinuity.
A: payoff for Ruth
B: "" Chris's
(a_{ij}, b_{ij})

0-sum: B = -A  \ a_{ij} + b_{ij} = 0

Given Chris uses strategy y, Ruth has payoff vector A_y
Optimal reply is any row where A_y is maximal
always a pure optimal reply by Ruth if y known
Theorem: If o som games $A, A'$ have $A_{ij} \leq A'_{ij}$ for all $ij$ then $Value \ of \ A \leq Value \ of \ A'$

[Bonus Problem]

Thm: Value is contin. func. of entries of A.
For 0-sum game, adding rows can only increase value;

Adding columns can only decrease the value.

Not so for general sum games.

e.g. Chicken:

\[
\begin{array}{c|cc}
 & D & S \\
\hline
P & (-M, -M) & (2, -1) \\
P & (-1, 2) & (1, 1) \\
\end{array}
\]

\((D, S), (S, D)\) are pure NE.

If \(P2\) uses \((p, 1-p)\) I get:

\[
D: -Mp + 2(1-p) = 2 - (M+2)p
\]

\[
S: -p + \frac{3}{4} - p = 1 - 2p
\]

Equal when \(1-2p = 2 - (M+2)p\) \(p = \frac{1}{M}\)
Outcome if both use \((\frac{1}{m}, 1-\frac{1}{m})\) is \(1-\frac{2}{M}\).

If \(M\) larger, \(-M\) smaller, outcome larger.
Outcome increases as game decreases.

Removing strategies can also help in general sum game.
Partial / Asymmetric Information

\[
\begin{bmatrix}
1 & 4 \\
0 & 3
\end{bmatrix} \quad \begin{bmatrix}
1 & 0 \\
4 & 1
\end{bmatrix}
\]

A \quad \hat{A}

Play A or \hat{A} based on coin toss.

If coin is known: Ruth pick Row 1 in A
Row 2 in \hat{A}

This is a saddle point, so same happens if Ruth declares a move first.

If coin unknown: \[
A + \hat{A} \cdot \frac{1}{2} = \begin{bmatrix}
1 & 2 \\
2 & 2
\end{bmatrix}
\]

Ruth: 2 \quad \text{Value 2}
Chris: 1

If Ruth sees the coin and makes optimal choice for selected game she gets 1 < 2