Recall: Nash equilibrium; strategies \((x, y)\) such that \(x\) is a best response to any \(y\) for \(x\).

**Thm** In a game with finitely many actions for each player there is (at least one) Nash equilibrium.

Can fail in games with \(\infty\) action space, e.g. choose larger integer.

**War + Peace**:

<table>
<thead>
<tr>
<th></th>
<th>(W)</th>
<th>(P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(W)</td>
<td>((-1, -1))</td>
<td>((0, -2))</td>
</tr>
<tr>
<td>(P)</td>
<td>((-2, 0))</td>
<td>((1, 1))</td>
</tr>
</tbody>
</table>

N.E.: \((W, W)\) \(\triangleright\) stable

\((P, P)\)

\((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\) unstable
If in \((W+P)\) opponent picks \((\frac{1}{3}, \frac{2}{3})\),

I get \(-\frac{1}{3}\) from \(W\).

0 from \(P\).

Best response is \(P\).

\(\Rightarrow\) Opponent also chooses \(P\).

\[\begin{array}{cc}
(1, 0) & (2, 1) \\
(0, 3) & (3, 2)
\end{array}\]

Iteration does not converge.

\((\frac{1}{2}, \frac{1}{2})\) is the unique N.E.
Brouwer's fixed point theorem

**Thm:** Let $K$ be a closed, bounded, convex set in $\mathbb{R}^n$, and $f : K \to K$ be continuous, then $f$ has a fixed point.

$\exists x \in K$ s.t. $f(x) = x$. 

If $K$ is not convex, rotation has no f.p.

$k = \text{circle}$, no interior.
If $K$ not bounded: $f(x) = x + 1$

$K = \mathbb{R}$

$K$ not closed: $K = (0, 1)$

$f(x) = \frac{x}{2}$
Given Strategies \((x, y)\) let

\[ f(xy) = (\hat{x}, \hat{y}) \]

where

\(\hat{x}\) is optimal against \(y\)

\[ \hat{y} = \frac{1}{\hat{x}} \]

\(x \in \Delta^m\), \(y \in \Delta^n\), \((xy) \in \Delta^m \times \Delta^n\)

\((\hat{x}, \hat{y}) \in \Delta^m \times \Delta^n\)

fixed point of \(f\) is a NE

\(K = \Delta^m \times \Delta^n\) closed bounded convex \(\checkmark\)

\(f\) not contin.

\(\checkmark\)

E.g. in W+t, optimal response to \((\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)\)

is \((0, 1)\)

response to \((\frac{1}{2} + \varepsilon, \frac{1}{2} - \varepsilon)\) is \((1, 0)\)
Goal: find $f: \Delta^m \times \Delta^m \to \Delta^m$

and fixed point is a N.E.

For given $xy$, let $x^*Ay$ is expected outcome for $P_i$. Call this $V_i(xy) = V_i$

for each action $i$, let $C_i = \text{benefit by switching to action } i$.

$c_i = (0 - 1 - 0) Ay - V_i$

If $c_i > 0$, $x$ is not optimal against $y$

Let $\hat{x}_i = \frac{X_i + C_i^+}{\sum_j X_j + C_j^+}$

$c^+ = \max(0, c)$

So $\sum \hat{x}_i = 1$

increase chance of beneficial choice.
\[ \hat{y}_i = \frac{y_i + d_i^+}{\sum_j y_j + d_j^+} \]

\[ d_i = \text{gain for } R_2, \text{ by switching to } i. \]

\[ f(x,y) = (\hat{x}, \hat{y}) \text{ is contin.} \]

By BFPT, \( \exists (x,y) \text{ s.t. } f(x,y) = (x,y) = (x,y) \)

However, this implies \( \hat{x} = x \) so \( c_i^+ = 0 \)

so \( xy \) is a N.E.
Proven in $d=1$:

$K = [a, b]$

$f : [a, b] \rightarrow [a, b]$ \hspace{1cm} \text{contin.}

Intermediate Value Theorem:

$f(a) \geq a$

$f(b) \leq b$

so $\exists x \text{ s.t. } f(x) = x$. 
Sperner's Lemma

There is a small triangle with all 3 colours. There is an odd number of such triangles.
In dim. \( n \):

Cut into smaller pyramids on each facet, use only the colours of its corners.

**Lemma:** \( \exists \) odd number of small simplices with all \( n+1 \) colours.

Odd # of segments with both colours.

Easy in \( n=1 \).
double counting.

Count $\rightarrow N$

Red-blue edge with a triangle.

Each red-blue edge is in 2 triangles, unless it is on the outside.

$$N = N_{RB}^0 + 2 N_{RB}^I = \text{odd}.$$
all other \( \Delta \)'s have no RB edge.

\[
N = 2N_{RBB} + 2N_{RRB} + N_{RBB}
\]

\( N \) odd, so number of red-blue-black

\( \Delta \) is also odd.