Solve a game: several meanings

1. find value
2. find some optimal strategy
3. "all"

eg. \( \begin{pmatrix} 4 & 2 \\ -6 & 2 \end{pmatrix} \) Value is 2 by S.P.

row 1 are optimal.

col. 2

If both used \((x_1, x_2)\) \(x^T A = (4x_1 - 6x_2, 2x_1 + 2x_2) \)

\(2\)

any \(x_1 > \frac{4}{5}\) is optimal.
when is a dominated strategy viable?

Let $S$ be set of columns that Chris will play with prob. $\neq 0$.

If a row has equality in all cols of $S$ and is dominated by another then it may be viable.

Given value $V$ of a game $A$, $x$ is optimal if and only if $x^T A \geq V$ in each entry.

$y$ optimal iff $A y \leq V$. 
\[
\begin{pmatrix}
0 & 3a \\
1 & 0 & 1 \\
a & 2 & 3
\end{pmatrix} = A
\]

If find \( x, y \) s.t. \( x^T A = (v, v, v) \)
\( A y = (v) \)
\( x_1 = 1 \)
\( x_2 = 1 \)
\( y = v A^{-1} (1) \)

This is only valid if solutions \( x, y \) are strategies.

e.g. \( A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \) Value is 1 (S.P.)

\[
A(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}) = (v) \quad A^{-1} = \begin{pmatrix} 1 & -2/3 \\ 0 & -1/3 \end{pmatrix}
\]

\[
A^{-1}(1) = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}
\]

\( A(\begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}) = A^T \)
\[
A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \quad 2 \text{ is S.P.}
\]

\[
A^{-1} = \frac{1}{5} \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}
\]

\[
A^{-1}(1) = \left( \frac{1}{5} \right) \circ \frac{V}{5} \Rightarrow A^{-1}(v) = \left( \frac{1}{5} \right)
\]

\[
(x_1, x_2) A = (v, v) \quad x = \Phi \left( \frac{5}{2} \frac{5}{2} \right) A^{-1} = \frac{5}{2} \frac{1}{5} (3, -1)
\]

\[
= \left( \frac{3}{2}, -\frac{1}{2} \right)
\]

Strategy of getting always same value is valid if it gives \( x_i \geq 0 \quad y_i \geq 0 \quad \forall i \).
If $x, y \in \Delta^m$, $y \in \Delta^n$

and $x^r A \geq v$ and $A y \leq v$

then $x, y$ are optimal.
A: payoff for Ruth
B: "" Chris

(c_{ij}, b_{ij}) Ruth picks i
Chris picks j

O-sum: \( a_{ij} + b_{ij} = 0 \) \( B = -A \)

safety: maximize worst case.

for Ruth: pick \( x \) s.t. \( \min_{y \in \Delta^n} x^T A y \)
as large as possible.

gives outcome max \( \min_{y \in \Delta^m} x^T A y \)

For O-sum game, if both players use safety strategies, get same outcome, = Value.
**War and Peace:**

\[
\begin{array}{c|c}
W & P \\
\hline
(-2, -2) & (-1, -3) \\
(1, 1) & (1, 1) \\
\end{array}
\]

\[
A = \begin{pmatrix} -2 & -1 \\ -3 & 1 \end{pmatrix}
\]

Safety strategy: War.

Nash equilibrium: x, y strategies s.t.

x is best response to y

y is a r x

(W, W) is N.E.
(P, P) is N.E.

**Indifference:** find y s.t. ac

if opponent use (\(\frac{2}{3}, \frac{1}{3}\))

W gives \(\frac{2}{3}(-2) + \frac{1}{3}(-1) = -\frac{5}{3}\)

P gives \(\frac{2}{3}(-3) + \frac{1}{3}(1) = -\frac{5}{3}\)

Both using (\(\frac{2}{3}, \frac{1}{3}\)) is N.E.
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K players.

Pick $s_1, \ldots, s_K$ in $[0, 1]$

Payoff to player $i$ is $s_i \left(1 - \sum_{j} s_j\right)^+$

$x^+ = \max(x, 0) = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$

e.g. if use $(0.1, 0.2, 0.3)$ then $\sum s_j = 0.6$

Payoff is $(0.1 \cdot 0.4, 0.2 \cdot 0.4, 0.3 \cdot 0.4) = (0.04, 0.08, 0.12)$

If use $(0.2, 0.4, 0.6)$ payoff is $(0, 0, 0)$

Pick $x, y, z$. Player 1 gets $x(1-x-y-z)^+$

$x(a-x)^+$ has maximum at $x = \frac{a}{2}$
If \( y_1, y_2 \) known, best \( x \) is \( \frac{1 - y_2}{2} \)

If \( s_1, \ldots, s_K \) is a N.E.,

then \( S_i = 1 - \frac{\sum_{j \neq i} s_j}{2} \)

\[
1 = \left( \sum_j s_j - s_i \right) + 2s_i = \left( \sum_j s_j \right) + s_i
\]

so all \( s_i \) are equal to \( \frac{1}{K+1} \)

payoff is \( \left( \frac{1}{K+1} \right)^2 \) to each.

If play s.t. \( \sum_j s_j = \frac{1}{2} \) payoff is \( \frac{1}{2} s_i \)

If \( s_i = \frac{1}{2K} \) for all \( i \), get \( \frac{1}{4K} \).
Thm. In any game with finite set of actions, $\exists$ Nash Equilibrium.