Recall: \( \text{mex}(S) = \min \text{ excluded integer} \).

\[ g(x) = \text{mex}(\{g(y) : x \rightarrow y\}) \]

\( y \) a follower of \( x \)

Thm: In \( G_1 + G_2 + \ldots + G_K \),

\[ g(x_1, x_2, \ldots, x_K) = g(x_1) \oplus g(x_2) \oplus \ldots \oplus g(x_K) \]

e.g. If subtraction set \( \{1, 2, 3, 4\} \)

then \( g(n) = (n \mod 5) \)

Thm: \( x \) is a \( p \)-pos. iff \( g(x) = 0 \).
\[\begin{array}{ccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
g(n) & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
\end{array}\]

\[g(n) = (n \mod 5)\]

\[(n \mod k) \text{ in } \{0, \ldots, k-1\}\]

\[x \equiv y \mod k : x-y \text{ divisible by } k\]

---

If subtract set is \([1, \ldots, k]\)

\[g(n) = (n \mod (k+1))\]

to change the Nim-sem from 5 to 0 need to replace \(x\) by \(x \oplus 5\)

\[4 \rightarrow 4 \oplus 6 = 2\quad 3 \oplus 6 = 5\]

\[2 \rightarrow 2 \oplus 6 = 4\]
Idea of proof:

Show that if \((x_1, \ldots, x_k)\) is a leader then there exists a follower with any value less than \(g(x_1) \oplus \cdots \oplus g(x_k)\), but no follower has \(g\)-value equal to any of those values.

By induction,

\[
  g(x_1, x_2, \ldots, y_i, \ldots, x_k) = g(x_1) \oplus g(x_2) \oplus \cdots \oplus g(y_i) \oplus \cdots \oplus g(x_k) \\
  \neq g(x_i)
\]

so

\[
  g(x_1, \ldots, y_i, \ldots, x_k) \neq g(x_1) \oplus \cdots \oplus g(x_k)
\]

Second claim: Similar to solution of NIM.
Asymmetric (Partisan) Games.

e.g. Subtraction game.

\[ S_A : \text{Alice can take any number from } S_A \]
\[ S_B : \text{Bob can take any number from } S_B \]

e.g. \[ S_A = \{1, 2, 3, 4\} \]
\[ S_B = \{1, 2, 3\} \]

If \( n = 5 \):
- If Alice to play:
  \[ 5 \xrightarrow{A} 4 \]
  \[ x \leftarrow B \]
  \[ 1, 2, 3 \xrightarrow{A} 0 \]

- If Bob to move:
  \[ 5 \xrightarrow{B} \{2, 3, 4\} \xrightarrow{A} 0 \]

Claim: Bob wins only if start at 1, 2, 3 and Alice move 2nd or start at 0, Alice 1st.
Precise assumption:

If in $\hat{G}$ $x \xrightarrow{B} y$ (Bob can move $x \rightarrow y$)
then also in $G$ $x \xrightarrow{B} y$

E.g. Subtraction with $\hat{S}_B \subset S_B$

If in $\hat{G}$ Bob has additional moves to $G$
then $\hat{G}$ is better for Bob than $G$
(Alice has same moves in both)

E.g. $S_A = \{1, 4\}$ $S_B = \{2, 3\}$

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>out</td>
<td>P</td>
<td>A</td>
<td>N</td>
<td>B</td>
<td>A</td>
<td>N</td>
<td>B</td>
<td>A</td>
<td>N</td>
</tr>
<tr>
<td>A0</td>
<td>P</td>
<td>P</td>
<td>A</td>
<td>N</td>
<td>A</td>
<td>N</td>
<td>P</td>
<td>A</td>
<td>N</td>
</tr>
<tr>
<td>B0</td>
<td>P</td>
<td>B</td>
<td>N</td>
<td>B</td>
<td>N</td>
<td>P</td>
<td>B</td>
<td>N</td>
<td>B</td>
</tr>
</tbody>
</table>
Possible Outcomes:

1. 1st player wins: N-pos.
2. 2nd " " P-pos.
   - Alice wins \( \rightarrow \) regardless of turn order.
   - Bob wins \( \rightarrow \)

Example:
\[ S_A = \{1, 2, 3, 4\} \quad S_B = \{1, 2, 3\} \]

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>at</td>
<td>P</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>A</td>
</tr>
</tbody>
</table>

**Thm:** If in game \( \hat{G} \) Alice has same moves as \( G \) but Bob has fewer \( \leq \) moves, then if Alice wins in \( G \), she also wins in \( \hat{G} \).

Removing moves from Bob's can only help Alice.
Define: \( P_A \) = win. pos. for Alice

\[ P_A = P-\text{pos} \text{ or } A-\text{pos} \]

\[ N_A = N-\text{pos} \text{ or } B-\text{pos} \]

Claim: If \( x \xrightarrow{A} y \) and \( y \in C \)

In non-partisan case, \( P_A = P_B = P-\text{pos} \). If \( x \rightarrow y \) and \( y \) is \( P-\text{pos} \), then \( x \) is \( N-\text{pos} \).

If all followers of \( x \) are \( N-\text{pos} \), then \( x \) is \( P-\text{pos} \).
Partisan rule:

- If $x \approx y$ and $y \in P_A$ then $x \in N_B$
- If all of Alice's moves from $x$ are in $N_A$ then $x \in P_B$

Same with $A; B$ swapped.
Go-moku: goal is 5-in-a-line

Move restriction + Strategy stealing:

1st player can win or draw

```
  X  X  O
   X  X
    O  O
```

```