Markov Chain Monte Carlo

Hard Core Model:

\[ \text{non overlapping spheres in a domain} \]

Idea! Sample a config by using a M.C.

possible M.C.: move a coin to a uniform valid location

Another M.C.: w.p. \( p \) add a coin if possible
\[ \begin{cases} 1-p & \text{remove a coin.} \end{cases} \]

Note: A stat. dist. has many M.C.'s

Triv. M.C.: \( P_{xy} = \pi_y \)
Metropolis M.C.

given a desired dist $\Pi$, construct some graph on the
state space

**Step of M.C.:** pick an edge $(x,y)$ [no matter how]

If $X_n = x$, pick e far from $x$: do nothing.

pick $e=(x,y)$: jump to $y$ w.p. $\left( \frac{\Pi_y}{\Pi_x} \wedge 1 \right)$

Stay at $x$ w.p. $1-\left( \frac{\Pi_y}{\Pi_x} \wedge 1 \right)$

$a \wedge b = \min(a,b)$

**Claim:** $\Pi$ is the stat dist for this M.C.

**Note:** Graph must be connected.

choice of Graph is an art.
Mixing time: How many steps before $X_n$ is close to $\Pi$?

**Proof of claim:** The M.C. is reversible w.r.t. $\Pi$.

Let $w_{xy} = P(\text{pick } xy)$

$$p_{xy} = w_{xy} \left( \frac{\Pi_y}{\Pi_x} \wedge 1 \right)$$

so 

$$\Pi_x p_{xy} = w_{xy} (\Pi_y \wedge \Pi_x)$$

same if swap $xy$. So $\Pi_x p_{xy} = \Pi_y p_{yx}$.

Hand Square model

Forbid

\[ \text{do not overlap.} \]
State space: \( \mathcal{V} = \text{vertices} \), all \( \sigma \in \{0,1\}^\mathcal{V} \) where if \( x, y \) then \( \sigma_x, \sigma_y \neq \text{both } 1 \).

dist.: for a parameter \( \lambda \), \( \Pi(\sigma) = \frac{\lambda^{|\sigma^0|}}{Z} \)

where \( Z = \sum_{\sigma} \lambda^{|\sigma^0|} \)

\( |\sigma^0| = \sum \sigma_x = \# \text{ particles} \)

If \( \sigma^0, \sigma^1 \) differ in one vertex \( \sigma_x^0 = 0, \sigma_x^1 = 1 \)

\( \Pi(\sigma^1) = \lambda \cdot \Pi(\sigma^0) \) since \( \sigma^1 \) has one more particle.

If \( \lambda > 1 \): Metropolis MC. pick uniform vertex \( x \).

If valid: add a particle to \( x \).

If already a particle at \( x \), remove it, w.p. \( \frac{1}{\lambda} \).
Observation: If $\lambda \gg 1$ then get long distance correlation.

If $\lambda$ not large enough: no correlation.

Thm: $\exists \lambda_c$ s.t. if $\lambda > \lambda_c$ \[ \text{Corr}(\sigma_x, \sigma_y) \geq C > 0 \]

If $\lambda < \lambda_c$ \[ |\text{Corr}(\sigma_x, \sigma_y)| \leq e^{-\alpha |x-y|} \]
Puzzle!

\[ x \quad 2x \]

Should you swap?

Obviously doesn't matter.

But: \[ \text{w.p. } \frac{1}{2} \quad 2x \quad [\text{cond. on seeing } x] \]

\[ \text{w.p. } \frac{1}{2} \quad \frac{x}{2} \]

So \[ E[\text{change} \mid \text{see } x] = 2x \cdot \frac{1}{2} + \frac{x}{2} \cdot \frac{1}{2} = \frac{5}{4}x \]