Recall: Joint distribution!

\[ p(x,y) = P(X=x, Y=y) \] discrete pmf

\[ f(x,y) = \frac{1}{\theta} P((X,Y) \in A) = \int_A f(x,y) \, dx \, dy \] contin. pdf

Marginal: \[ p_X = \text{pmf of } X \]

\[ f_X = \text{pdf of } X \]

\[ f_X(x) = \int_{-\infty}^{\infty} f(x,y) \, dy \]

\[ f_Y(y) = \int_{-\infty}^{\infty} f(x,y) \, dx \]

Note: can have contin. X, Y s.t. (X, Y) not contin but each of X, Y is contin.

e.g. \((X, Y) = \text{unit pt on diagonal } (0,0) \rightarrow (1,1)\)
There is no \( f \) s.t. \( p(x,y) = \int_A f(x,y) \, dx \, dy \).

For jointly contin \((XY)\), \( p(x=y) = 0 \)

Note: a R.V. \( X \) is contin iff CDF \( F_X \) is differentiable.

PDF can have jumps.

(Ex.: Unif[0,1])

Thm: \( E(X+Y) = (E(X) + E(Y)) \) assuming defined.

E.g.: If \( X \) = # Black cards in sample of size \( n \), \( E(X) = \frac{n}{2} \)

E.g. 3-birthday collision: \( n \) people, \( \binom{3}{1} \) triples, each is a collision w.p. \( \frac{1}{365^2} \)
$X = \# \text{ of triple collisions, then } EX = \frac{1}{365^2} \cdot \binom{n}{3}$

$EX = \frac{n(n-1)(n-2)}{6 \cdot 365^2}$

Poisson heuristic: If there are unlikely events, but $E(\# \text{ events that occur})$ is $\lambda$ then $\# \text{ of occurrences is}$

$\approx \text{ Pois} (\lambda)$

$P(\text{no triple collis.}) \approx e^{-\lambda}$  \quad $\lambda = \binom{n}{3} \cdot \frac{1}{365^2}$
Independent RV:

**Definition:** R.V. $X, Y$ are independent if for any $A, B \subset \mathbb{R}$,

$$P(X \in A, Y \in B) = P(X \in A) P(Y \in B)$$

**Intuition:** Knowing $X$ does not tell us anything about $Y$, and vice versa.

This is enough to show

$$P(X \leq s, Y \leq t) = F_{XY}(s,t) = F_X(s) F_Y(t) \quad \forall s, t$$

where $F_{XY}(s,t)$ is the joint CDF.
For disc. RV. $XY$:

Independence iff $p(x,y) = p_X(x) \cdot p_Y(y)$

For contin. $f$:

Independence iff $f(x,y) = f_X(x) \cdot f_Y(y)$.

e.g. if $X, Y$ have pdf $2e^{-x-y}$ on $[0,\infty) \times [0,\infty)$

$2e^{-x-2y} = e^{-x} \cdot 2e^{-2y}$

$X, Y$ are independent.

On $[0,1] \times [0,1]$ $f_{XY} = x + 2y^3$

not a product of $f_X(x), f_Y(y)$ so not independent.
Idea of proof: \( P(\{X \in A, Y \in B\}) = \iiint_{A \times B} f(x, y) \, dx \, dy \)

If \( f(x, y) \) is a product, this factors

If independent, then use \( F(x, y) = F_X(x) \cdot F_Y(y) \)

at 4 points: \((x, y), (x+\epsilon, y), (x, y+\epsilon), (x+\epsilon, y+\epsilon)\)

Find \( P(x, y \in \square) \)

\( \det \epsilon \to 0 \)
Let $X$ be $\exp(1)$.
Given $X$, let $Y$ be unif $[0,X]$.

What is the pdf?

**Thm:** If $X,Y$ contin. Joint pdf is $\frac{d}{dx} \frac{d}{dy} F(x,y)$

where $F(x,y) = P(X \leq x, Y \leq y)$

In the example:

![Diagram of a probability density function (pdf) with shaded regions for $F(x,y)$ and points along the x and y axes.](image)
If \( y > x \):

\[ y \geq x \Rightarrow F(xy) = P(X \leq x) = 1 - e^{-x} \]

\[ \frac{d}{dx} \frac{d}{dy} = 0 \text{ when } y > x \text{ as expected.} \]

If \( y < x \):

\[ F(xy) = P(X \leq y) + P(X \in [y, x], Y \leq y) \]

\[ = 1 - e^{-y} + \int_y^x e^{-t} dt \cdot \frac{y}{t} \]

Take \( \frac{d}{dx} \frac{d}{dy} \): \[ f(xy) = \frac{d}{dx} \frac{d}{dy} \int_y^x e^{-t} dt \cdot \frac{y}{t} \]
\[ f(x, y) = \frac{d}{dy} \left( e^{-x} \frac{y}{x} \right) = \frac{e^{-x}}{x} \]

**sol.** \( f(x, y) = \frac{e^{-x}}{x} \) on wedge \( 0 \leq y \leq x \) \\
\( 0 \) elsewhere

not a factor since

\[ f(x, y) = \frac{e^{-x}}{x}, 1 < x \geq 0, 1 > y > 0, 1 y \leq x \]
More details: If \( y \leq x \) then \( P(x \leq x, y \leq y) = P(A) + P(B) \)

\[
P(A) = P(x \leq y) = 1 - e^{-y}
\]

\[
P(B) = \int_{y}^{x} p(x = t) dt \quad P(y \leq y | x = t)
\]

\[
= \int_{y}^{x} e^{-t} dt \frac{y}{t}
\]

and so \( \frac{d}{dx} \frac{d}{dy} [P(A) + P(B)] = \frac{e^{-x}}{x} \)

\( \Box \) this is a slight abuse of notation, since \( P(x = t) = 0 \).

However, this can be justified

If \( x = t \), \( y \) is uniform on \([0, t]\) so \( P(Y \leq y) = \frac{y}{t} \).
Law of uncon. statistician: \[ E(g(x,y)) = \iiint g(x,y) f(x,y) \, dx \, dy \]

If \( g(x,y) = xy \), and \( x, y \) indep:

\[
E(xy) = \iint xy \, f(x) \, f_y(y) \, dx \, dy = \left( \int x \, f_x(x) \, dx \right) \left( \int y \, f_y(y) \, dy \right) = E(X) \cdot E(Y)
\]

Thm: If \( x, y \) indep. then \( E(xy) = (E(x))(E(y)) \).