Recall that the probability density function (pdf) of a continuous RV is $f(x)$ on $\mathbb{R}$ if

$$P(X \in [a,b]) = \int_a^b f(x) \, dx.$$ 

Must have $f \geq 0$, $\int_{-\infty}^{\infty} f(x) \, dx = 1$.

e.g. Uniform on $[a,b]$:

$$f(x) = \left\{ \begin{array}{ll}
\frac{1}{b-a} & x \in [a,b] \\
0 & \text{otherwise}
\end{array} \right.$$ 

Exponential with parameter $\lambda$:

$$f(x) = \left\{ \begin{array}{ll}
\lambda e^{-\lambda x} & x > 0 \\
0 & x < 0
\end{array} \right.$$ 

$$P(X > a) = \int_a^{\infty} \lambda e^{-\lambda x} \, dx = e^{-\lambda a}.$$
Memoryless Exponential

Given $X \geq a$, what is $X - a$?

What is

$$P(X - a \geq t \mid X \geq a) = \frac{P(X \geq a, X \geq a + t)}{P(X \geq a)}$$

$$= \frac{P(X \geq a + t)}{P(X \geq a)}$$

$$= \frac{e^{-\lambda(a + t)}}{e^{-\lambda a}} = e^{-\lambda t}$$

Cond. on $X \geq a$, $X - a$ is again Exp with param $\lambda$.

Compare to Geom$(p)$: has

$$P(X = n) = p (1 - p)^{n-1}$$

$$P(X > n) = (1 - p)^n = e^{n \log(1 - p)}$$

$$P(X > n + m \mid X > n) = P(X > m)$$
Gaussian / Normal

two parameters: μ, σ
μ = mean
σ = std. dev. > 0

pdf of $N(μ, σ^2)$ is

$$f(x) = \frac{1}{\sqrt{2πσ^2}} e^{-\frac{(x-μ)^2}{2σ^2}}$$

Standard Normal has $μ = 0$, $σ = 1$

Claim: $\int_{-∞}^{∞} e^{-x^2/2} = \sqrt{2π}$

PF: $I^2 = \left( \int_{-∞}^{∞} e^{-x^2/2} dx \right) \left( \int_{-∞}^{∞} e^{-y^2/2} dy \right) = \iint e^{-\frac{(x+y)^2}{2}} \, dx \, dy$

$$= \int_{0}^{2π} \int_{0}^{∞} r e^{-r^2/2} \, dr \, dθ = \int_{0}^{2π} \left[ -e^{-r^2/2} \right]_0^∞ \, dθ = 2π$$
Let \( X \) has pdf \( f \).

Let \( Y = X + \mu \) . **Claim:** \( Y \) has pdf \( g(t) = f(t - \mu) \)

**Proof:** 
\[ P(X \in [a,b]) = P(Y \in [\mu + a, \mu + b]) \]
Is satisfied by \( f, g \) as above by co.v.

Let \( Y = a \cdot X \). 

\[ g(t) = \frac{1}{a} f\left(\frac{t}{a}\right) \]

\[ g(at) = \frac{1}{a} f(t) \]

\[ g\left(\frac{t}{a}\right) = a \cdot f(t) \]

**Proof:** 
\[ P(X \in [a,u]) = P(Y \in [au, au + v]) \]
Claim: \( N(\mu, \sigma^2) \) is \( \mu + \sigma \cdot N(0,1) \)

Central Limit Theorems

A sum of many roughly independent things is \( \approx \) Normal.

\( \mu \) = most likely value, mean, "typical"

\( \sigma \) = order of fluctuations

CDF of normal: \( P(X \leq t) = \int_{-\infty}^{t} f(x) \, dx \)

If \( X \) is \( N(0,1) \): \( P(X \leq t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \, e^{-x^2/2} \, dx \)

called Error function (erf) \( \Phi \)
For $N(\mu, \sigma^2) = Y$ \quad $Y = \mu + \sigma X$ \quad where $X$ is $N(0,1)$

\[
P(Y \leq t) = P(\sigma X + \mu \leq t) = P(X \leq \frac{t-\mu}{\sigma}) = F\left(\frac{t-\mu}{\sigma}\right)
\]