Problem 1. A particle moving in two dimensions performs a random walk as in class: The particle starts at \((0, 0)\) and makes successive steps of length 1 that are mutually independent, with each step either north, south, east or west with equal probabilities \(\frac{1}{4}\). Let \(S_n \in \mathbb{Z}^2\) denote the position of the particle after \(n\) steps, so \(S_n\) is the point in the integer lattice \(\mathbb{Z}^2\) given by the vector sum \(X_1 + \cdots + X_n\) of the first \(n\) steps. Thus \(S_n\) is a random vector.

(a) The expectation of a random vector \(S = (X, Y)\) is given by \(\mathbb{E}S = (\mathbb{E}X, \mathbb{E}Y)\) and the variance is given by \(\text{Var}(S) = \mathbb{E}(S - \mathbb{E}S) \cdot (S - \mathbb{E}S)\), with the dot indicating the dot product. Show that, for the random walk, \(\mathbb{E}S_n = (0, 0)\) and \(\text{Var}(S_n) = n\).

(b) If the random walk is instead 1-dimensional (probability \(\frac{1}{2}\) of steps left or right) or 3-dimensional (probability \(\frac{1}{6}\) of steps north, south, east, west, up, down), what is the expected position and the variance of the walk after \(n\) steps?

Problem 2. Consider simple symmetric random walk in two dimensions.

(a) Show that the probability \(p_{2n}\) that a walker returns to its starting place at the origin after \(2n\) steps is given by \(p_{2n} = \left(\frac{1}{4}\right)^{2n} \sum_{k=0}^{n} \binom{n}{k}^2\).

(b) Show that \(\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}\). Hint: Think of choosing \(n\) balls from a box that has 2\(n\) balls, \(n\) white and \(n\) black.

(c) Conclude that \(p_{2n} \sim \frac{1}{\pi n}\) as \(n \to \infty\).

(d) Conclude that the two-dimensional walk is recurrent.

Problem 3. (a) Consider a walk in \(\mathbb{Z}\) that takes steps 0 or \(\pm 2\), such that the probability of step 2 equals the probability of step \(-2\). Show that this walk is recurrent. (Hint: use the fact that the simple random walk on \(\mathbb{Z}\) is recurrent.)

(b) Use this to show that if \(X_n\) and \(Y_n\) are two random walks in \(\mathbb{Z}\) with step 1 with probability \(p\) and \(-1\) with probability \(1 - p\), then there are infinitely many \(n\)'s for which \(X_n = Y_n\). (Hint: what is \(X_n - Y_n\)?)

Problem 4. Let \(X\) be the number of queens in a 13 card hand out of a standard deck, and let \(Y\) be the number of kings. What is \(\mathbb{E}(X|Y)\)?

Problem 5. Let \(X\) be uniform in \(\{0, 1, 2, \ldots, 8\}\). We put \(X\) black balls and \(8 - X\) white balls in a bag. Now sample 10 balls out of the bag (returning each ball before sampling the next). Let \(Y\) be the number of black balls out of the 10 samples. What is \(\mathbb{E}(Y|X)\)? What is \(\mathbb{E}(X|Y)\)?

Problem 6. This problem is concerned with random walks in \(\mathbb{Z}^2\) which take steps equal to each of the four vectors of length 1 with probability \(\frac{1}{4}\) each, and is \(\mathbb{Z}^3\) with the probability \(\frac{1}{6}\) for each of the six unit vectors. As we did for 1-dimensional random walk: For each of those, as well as in 1-dimension, simulate 100 independent random walks starting at 0, each for up to \(10^6\) steps. For each walk, let \(T\) be the first time the walk returns to 0. If a walk does not return to 0, let \(T = 10^6\).

(a) In each of the three cases, how many walks did not return to 0?

(b) Find the sample mean of the \(T\)'s for the walks that did return to the origin in each case, and prepare a histogram of those \(T\)'s.
(c) In each dimension, for each of the 100 random walks, compute $\|S_{10^6}\|^2$. Make a histogram of those values, and compute the sample mean of those.

Additional problems:
(a) Consider i.i.d. integer random variables $X_n$ with $\mathbb{E}X_n = 0$ and $\text{Var}(X_n) < \infty$. We show that the random walk with these steps is recurrent: Let $\phi$ be the characteristic function of $X_i$. We’ve seen in class that

$$
\mathbb{E}N = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1 - \phi(t)} \, dt.
$$

Show that this is integrable near 0, and then that the integral is finite, and thus the random walk is recurrent. **note:** in fact, it is sufficient that $\mathbb{E}X_n$ exists and is 0 for the random walk to be recurrent.

(b) Show that the random walk on $\mathbb{Z}^3$ taking steps $(\pm 1, \pm 1, \pm 1)$ with probability $\frac{1}{8}$ each is transient. Show that the characteristic function of its steps is $\phi(t) = \cos(t_1) \cos(t_2) \cos(t_3)$. 