Problem 1. Let $Z$ be a standard normal variable. Find all the moments of $Z$. (Hint: expand the characteristic function as a Taylor series.)

Solution. The characteristic function of $Z$ is $\phi(t) = e^{-t^2/2}$. This has Taylor series

$$\phi(t) = \sum_{n=0}^{\infty} \frac{(-t^2/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n^2} t^{2n}.$$\[1.5ex]

However, we know the coefficient of $t^n$ is $i^n E[Z^n]/n!$. Therefore if $n$ is odd then $E[Z^n] = 0$ (also follows since $Z$ is symmetric). For even moments, if $m = 2n$ we have $E[Z^m] = \frac{m!}{2^n n!}$.


Solution. Use $(X + Y)^3 = X^3 + 3X^2Y + 2XY^2 + Y^3$, and take expectation of each term. Since $X, Y$ are independent and $E[X] = E[Y] = 0$, this gives

$$E(X + Y)^3 = E[X^3] + E[Y^3].$$\[1.5ex]

Similarly, use $(X + Y)^4 = X^4 + 4X^3Y + 6X^2Y^2 + 4XY^3 + Y^4$ to find


Problem 3. Let $A$ be an $n \times n$ matrix, and let $X = (X_1, \ldots, X_n)^T$ be a vector of i.i.d. $N(0, 1)$ random variables. Let $Y = AX$, and $Y_i$ its coordinates.

(a) What is the distribution of $Y_i$?
(b) Find Cov($Y_i, Y_j$).
(c) If $n = 2$ and $A$ is invertible, show that $Y_1, Y_2$ have joint probability density function

$$\frac{1}{(2\pi)^{n/2} |\det A|} e^{-\frac{(y^T A^{-1} y)^2}{2}}$$

on $\mathbb{R}^2$, where $y = (y_1, y_2)^T$ is a vector.) (Hint: What is the Jacobian of the mapping from $X$ to $Y$?)
(d) If $A$ is invertible, show that $Y$ has probability density function with the same formula.

Solution. (a) $Y_i = \sum_k A_{ik}X_k$ is a sum of independent normal variables, so has distribution $N(0, \sum_k A_{ik}^2)$.
(b) Using the sums for $Y_i$ and $Y_j$ we get

$$\text{Cov}(Y_i, Y_j) = \sum_{k, \ell} \text{Cov}(A_{ik}X_k, A_{j\ell}X_\ell)$$

$$= \sum_{k, \ell} A_{ik}A_{j\ell} \text{Cov}(X_k, X_\ell)$$

$$= \sum_k A_{ik}A_{jk},$$

since the covariance is 0 unless $\ell = k$ and 1 if it is.
(c) The Jacobian of the map $Y = AX$ is $|\det A|$. We have $X = A^{-1}Y$, and so
$$\sum X_i^2 = X^TX = Y^TBY$$ with $B = (A^{-1})^T A^{-1}$. The joint density of $X$ is
$$\frac{1}{(2\pi)^{n/2}} e^{-X^TX/2},$$ and so the change of variable formula gives the claimed density of $Y$.

**Problem 4.** (a) If $X$ is an integer valued random variable, show that the characteristic function of $X$ has period $2\pi$.

(b) Prove the converse: if $\phi(t) = \phi(t + 2\pi)$ for every $t$, then show that $X$ takes only integer values. (Hint: If a random variable $Y$ satisfies $Y \geq 0$ and $E[Y] = 0$ then $P(Y = 0) = 1$. Use this for a carefully chosen function of $X$.)

**Solution.** (a) Since $X$ is an integer, $e^{2\pi i X} = 1$. Therefore, $e^{itX} = e^{i(t+2\pi)X}$, and so they have the same expectation.

(b) Note $\phi(2\pi) = \phi(0) = 1$, so $Ee^{2\pi iX} = 1$. To deduce that $X$ must be an integer, we use the hint. Considering the real part, we have $E \cos(2\pi X) = 1$, but $\cos(2\pi X) \leq 1$. The only way for a random variable that is always at most 1 to have expectation 1 is if it is always equal to 1. Therefore $\cos(2\pi X) = 1$ and so $X$ is an integer.

**Problem 5.** This problem is concerned with the random walk in $\mathbb{Z}$. Let $X_i = \pm 1$ with probability $\frac{1}{2}$ each be independent. Let $S_n = X_1 + X_2 + \cdots + X_n$.

(a) Simulate a random walk with $10^6$ steps. Submit your code and a plot of $S_0, \ldots, S_n$.

(b) Simulate 1000 independent random walks, each for up to $10^6$ steps. For each walk, let $T$ be the first time the walk returns to 0. If a walk does not return to 0, let $T = 10^6$. How many of the 1000 walks did not return to 0? Submit a histogram of the values of $T$ observed.

(c) Make a log-log plot of the fraction of times $T > k$ for $k = 0, \ldots, 10^6$.

(d) Based on the previous plot, guess the asymptotics of $P(T > n)$ as $n \to \infty$. What do you think is the mean of $T$?

**Solution.** (a) Sample graph:

(b) Sample histogram (up to 40):

(c) Sample plot:

(d) The plot suggests that $P(T \geq n) \approx c/\sqrt{n}$. (Compare with $1/\sqrt{n}$ which has slope $-1/2$ in a log-log plot.) This implies that $E[T] = \infty$. In fact, for this random walk it can be shown that
$$P(T = 2n) = \frac{2^{-2n}}{2n - 1} \binom{2n}{n} \approx \frac{n^{-3/2}}{2\sqrt{\pi}}.$$

Sample code:

```python
import numpy as np
import matplotlib.pyplot as plt

def get_RW(n):
    pass
```
X = np.random.random_integers(0,1,n)
X = 2*X-1
return np.cumsum(X)

# 6a

S = get_RW(1000000)
plt.figure()
plt.plot(S)
plt.savefig('hw7_q6a.pdf')

# 6b
def find_T(n):
    S = get_RW(n)
    for i in range(1,n,2):  # only odd i can have S[i]=0.
        if S[i]==0: return 1+i
    return n

T_list = [find_T(1000000) for i in range(1000)]
H = [T_list.count(i) for i in range(40)]

plt.figure()
plt.bar(range(40), H, 1, align='center')
plt.savefig('hw7_q6b.pdf')

# 6c

T_list_large = [x for x in T_list if x>=1000]
R = [sum(x>=n for x in T_list) for n in range(1000)]
R += [sum(x>=n for x in T_list_large) for n in range(1000,max(T_list))]
R = np.array(R)
X = np.arange(max(T_list))

plt.figure()
plt.axes().set_aspect('equal')
plt.plot(np.log(2+X),np.log(R/1000))
plt.plot([0,10],[0,-5])
plt.savefig('hw7_q6c.pdf')