Problem 1. Let $Z$ be a standard normal variable. Find all the moments of $Z$. (Hint: expand the characteristic function as a taylor series.)


Problem 3. Let $A$ be an $n \times n$ matrix, and let $X = (X_1, \ldots, X_n)^T$ be a vector of i.i.d. $N(0, 1)$ random variables. Let $Y = AX$, and $Y_i$ its coordinates.
   (a) What is the distribution of $Y_i$?
   (b) Find Cov($Y_i, Y_j$).
   (c) If $n = 2$ and $A$ is invertible, show that $Y_1, Y_2$ have joint probability density function
      \[ \frac{1}{(2\pi)^{n/2} |\det A|} e^{-(y^T A^{-2} y)/2} \]
      on $\mathbb{R}^2$, where $y = (y_1, y_2)^T$ is a vector.) (Hint: What is the Jacobian of the mapping from $X$ to $Y$?)
   (d) If $A$ is invertible, show that $Y$ has probability density function with the same formula.

Problem 4. (a) If $X$ is an integer valued random variable, show that the characteristic function of $X$ has period $2\pi$.
   (b) Prove the converse: if $\phi(t) = \phi(t + 2\pi)$ for every $t$, then show that $X$ takes only integer values. (Hint: If a random variable $Y$ satisfies $Y \geq 0$ and $E[Y] = 0$ then $P(Y = 0) = 1$. Use this for a carefully chosen function of $X$.)

Problem 5. This problem is concerned with the random walk in $\mathbb{Z}$. Let $X_i = \pm 1$ with probability $\frac{1}{2}$ each be independent. Let $S_n = X_1 + X_2 + \cdots + X_n$.
   (a) Simulate a random walk with $10^6$ steps. Submit your code and a plot of $S_0, \ldots, S_n$.
   (b) Simulate 1000 independent random walks, each for up to $10^6$ steps. For each walk, let $T$ be the first time the walk returns to 0. If a walk does not return to 0, let $T = 10^6$. How many of the 1000 walks did not return to 0? Submit a histogram of the values of $T$ observed.
   (c) Make a log-log plot of the fraction of times $T > k$ for $k = 0, \ldots, 10^6$.
   (d) Based on the previous plot, guess the asymptotics of $P(T > n)$ as $n \to \infty$. What do you think is the mean of $T$?