Problem 1. Let \( Z_1, \ldots, Z_n \) be i.i.d. standard normal random variables, and define \( Y = \sum_{i=1}^{n} Z_i^2 \). (\( Y \) is said to have a chi-squared distribution with \( n \) degrees of freedom; it is often used in statistics.)

(a) Find the characteristic function of \( Y \).

(b) In particular, show that if \( n = 2 \) then \( Y \) has an exponential distribution, and give the parameter.

Solution. (a) We first find the characteristic function of \( Z^2 \). This is

\[
E e^{itZ^2} = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} e^{itz^2} dz = \frac{1}{\sqrt{1-2it}}.
\]

(By completion of the square, as in class.) If we take a sum of \( n \) of those, we find

\[
\phi_Y(t) = (1 - 2it)^{-n/2}.
\]

(b) For \( n = 2 \), we have \( \phi(t) = (1 - 2it)^{-1} \), which is the characteristic function of a \( \text{Exp}(1/2) \) random variable.

Problem 2. The standard Cauchy random variable has probability density function \( f(x) = \frac{1}{\pi (1 + x^2)} \) and characteristic function \( \phi(t) = E[e^{itX}] = e^{-|t|} \). Suppose that \( X_1, X_2, \ldots \) are i.i.d. Cauchy and let \( S_n = \sum_{i=1}^{n} X_i \).

(a) We have seen that \( EX_1 \) is undefined. Check this via the characteristic function.

(b) Use characteristic functions to show that \( n^{-1}S_n \) is also a standard Cauchy random variable.

Solution. (a) We know that \( \phi'(0) = iE[X] \) when the expectation exists. However, \( e^{-|t|} \) is not differentiable at 0.

(b) As a sum of independent terms,

\[
\phi_{S_n}(t) = (e^{-|t|})^n.
\]

Dividing by \( n \) has the effect of replacing \( t \) by \( t/n \):

\[
\phi_{S_n/n}(t) = (e^{-|t/n|})^n = e^{-|t|}.
\]

Thus the average of \( n \) independent Cauchy random variables is also Cauchy.

Problem 3. If \( X, Y \) are independent standard normal random variables, we wish to show that \( X/Y \) has the Cauchy distribution.

(a) Use a change of variable to write \( X, Y \) in polar coordinates \( R, \Theta \), and show that \( \Theta \) is uniform on \([0, 2\pi]\).

(b) Deduce that \( X/Y \) has the Cauchy distribution.

Solution. (a) The joint density of \((X,Y)\) is \( \frac{1}{2\pi} e^{-(x^2+y^2)/2} \). By a change of variable \( X = R \cos(\Theta) \) and \( Y = R \sin(\Theta) \), with Jacobian \( R \) we find \( R, \Theta \) have joint pdf \( \frac{R}{2\pi} e^{-R^2/2} \). In particular, this is the product of \( 1/2\pi \) (the PDF of \( \Theta \)), and of \( Re^{-R^2/2} \) (the PDF of \( R \)).
(b) $X/Y = \cot \Theta$, and since $\Theta$ is uniform, it $\cot$ is the Cauchy random variable.

Problem 4. (a) Suppose that $X_1, \ldots, X_n$ are independent Gaussian random variables, with $X_i$ having $N(\mu_i, \sigma_i^2)$ distribution. Let $S_n = X_1 + \cdots + X_n$. Compute the characteristic function of $S_n$ and thereby identify its distribution.

(b) A fisherman catches four salmon. Their weights are independent $N(5, 4)$ random variables (this is an approximation; in reality the weights cannot be negative).

Find the probability that the last fish weighs more than the other three together.

(Hint: Consider $X_1 + X_2 + X_3 - X_4$. What is the distribution of $-X_4$?)

(c) What is the probability that some fish weighs more than the other three combined?

Solution. (a) The characteristic function of $X_i$ is $\phi(t) = e^{i\mu_i t} e^{-\sigma_i^2 t^2 / 2}$. Since they are independent, we have that $\phi_{S_n} = e^{iat - bt^2 / 2}$, where $a = \sum \mu_i$ and $b = \sum \sigma_i^2$.

Therefore $S_n$ is normal with mean $\sum \mu_i$ and variance $\sum \sigma_i^2$.

(b) We have that $X_1 + X_2 + X_3 - X_4$ is $Y = N(10, 16)$. We can write $Y = 10 + 4Z$, where $Z = N(0, 1)$. Therefore

$$P(Y < 0) = P(Z < -2.5) = \Phi(-2.5) = 0.0062.$$

(c) Let $A_i$ be the event that the $i$'th fish is heavier than all others together. Since these are disjoint, and each has probability $\Phi(-2.5)$, the probability of the union is $4\Phi(-2.5) = 0.0248$.

Problem 5. Let $X = \text{Geom}(p)$. Compute the characteristic function of $X$ and use it to show that $\text{Var}(X) = \frac{1-p}{p^2}$.

Solution. We have

$$\phi(t) = E[e^{itX}] = \sum_{n=1}^{\infty} p(1-p)^{n-1} e^{itn}$$

$$= pe^{it} \sum_{n=0}^{\infty} ((1-p)e^{it})^n$$

$$= \frac{pe^{it}}{1 - (1-p)e^{it}}.$$

Using this we find

$$\phi'(t) = \frac{ipe^{it}}{(1 - (1-p)e^{it})^2},$$

and

$$\phi''(t) = \frac{-pe^{it}(1 + (1-p)e^{it})}{(1 - (1-p)e^{it})^3}.$$

Therefore $\phi'(0) = i/p$ and $\phi''(0) = 1/p - 2/p^2$.

From this we find $E[X] = 1/p$ and $E[X^2] = 2/p^2 - 1/p$, and so $\text{Var}(X) = 1/p^2 - 1/p$. 

MATH 318 – Homework 6 solutions