Problem 1. A government wants to find out how many of its citizens are filling out fraudulent tax returns. It creates a survey with the single question “Do you fill out your tax return honestly?” Along with the question come the following Instructions: Toss a fair coin. If the result of the coin toss is heads, answer the question truthfully. If the result of the coin toss is tails, answer the question with NO regardless of whether you cheat or not.

Randomising the response makes it impossible to deduce for any single individual that they cheat. The result of the survey is that 45% of the respondents say YES. Suppose we interpret this as meaning that a randomly chosen member of the population will answer the survey YES with probability 0.45. Find the proportion of the population that fills out fraudulent returns.

Problem 2. Consider $n$ flips of a coin. A run is a sequence of consecutive tosses with the same result. For $k < n$, let $E_k$ be the event that a run is completed at time $k$; this means that the results of the $k^{th}$ and $(k+1)^{st}$ flips are different. For example, if $n = 10$ and the outcomes of the first 10 flips are HHHTTHTHTH then runs are completed at times 3, 5, 7, 9.

(a) Show that if the coin is fair, then the events $E_k$, $1 \leq k \leq n$ are independent. (This requires you to show that $P(\bigcap_{i=1}^{m} E_{k_i}) = \prod_{i=1}^{m} P(E_{k_i})$ for every choice of $m \leq n$ and $k_1 < \ldots < k_m$.)

(b) Show that if the coin is not fair, the events are not independent. (An unfair coin gives H with probability $p \neq \frac{1}{2}$.)

Problem 3. Eye colour is determined by a single gene, and each individual has two copies of the gene. For simplicity, we assume there are only blue and brown eyes. If both genes of an individual are blue, they have blue eyes; Otherwise they have brown eyes. (In technical terms, blue eyes are recessive and brown eyes are dominant.) A child inherits randomly one copy of the gene from each parent. (See e.g. season 4 of The bridge for the significance of these facts.)

(a) Suppose Fanny and her parents have brown eyes but her brother Alexander has blue eyes. What is the conditional probability that Fanny has two brown genes?

(b) Suppose moreover that Fanny has 3 children with Lars, who has blue eyes, and all 3 have brown eyes. What is the conditional probability of the same event?

Problem 4. Two hockey teams, A and B play a series of games, until one of the teams wins 4 games. Suppose team A has probability $p$ of winning each game, and games are independent. Let $X$ be the total number of games that are played.

(a) Find the probability mass function of $X$.

(b) What is the probability that team A wins the series conditioned on $X = 4$?

(c) What is the probability that team A wins the series conditioned on $X = 7$? (Simplify your expressions as much as possible.)

Problem 5. We wish to select between two options $A, B$ with probability $\frac{1}{2}$ each. We are given a coin which comes up heads with an unknown probability $p$. Show that the
following procedure works: Toss the coin twice. If the results are \((H, T)\) pick \(A\). If the results are \((T, H)\) pick \(B\). Otherwise start again.

**Problem 6.** A fair die is rolled four times.
(a) Let \(Y\) denote the number of distinct rolls. Find the probability mass function of \(Y\).
(b) Let \(Z\) denote the minimal result of the 4 throws. Find the probability mass function of \(Z\).

**Problem 7.** Coding component:
(a) Simulate 100,000 geometric random variables with parameter \(p = 0.01\) and create a histogram of the resulting values, with buckets for each of the values 1 to 1000.
(b) Next, create a plot of the probability mass function of the same geometric random variable, over the integers 1 to 1000. Briefly describe how this plot compares to the histogram from part (a).
(c) Finally, plot the function \(f(t) = e^{-t}\) for \(t\) between 0 and 10. Briefly compare this plot to the two plots above.
Submit your code, plots, and written answers to the questions in (b),(c). (Depending on your computer, the plots may take a few minutes to appear.)

**Extra practice problems**
Chapter 1: 12, 13, 19, 20, 23, 25, 30, 33, 40.