**Problem 1.** Find the stationary distribution (if it exists) of the following birth and death chains, with states are \{0, 1, \ldots\}, and in all cases, \( P_{0,-1} = 0 \) and \( P_{01} = 1 \).

(a) \( P_{i,i+1} = \frac{1}{i+1} \) and \( P_{i,i-1} = \frac{i}{i+1} \).

(b) \( P_{i,i+1} = \frac{1}{i+1} \) and \( P_{i,i-1} = \frac{1}{i+1} \).

\[ 0 = [1]/2 \quad [i-1] \quad i-1/i = [i] /i+1 \]

**Solution.**

(a) Detailed balance gives \( \pi_i = \frac{i+1}{i} \pi_{i-1} \), from which we get \( \pi_n = \frac{n+1}{n!} \pi_0 \).

Since

\[ 1 = \sum \pi_n = \sum \frac{n+1}{n!} \pi_0 = 2e \pi_0, \]

we have \( \pi_0 = \frac{1}{2e} \) and \( \pi_n = \frac{n+1}{2en!} \).

(b) Here \( \pi_i = \frac{(i-1)(i+1)}{i} \pi_i \), so \( \pi_i \) is increasing in \( i \) and cannot add up to 1, so there is no stationary distribution.

**Problem 2.** Tlaloc has 4 umbrellas, each either at home or at work. Each time he goes to work or back, it rains with probability \( p \), independently of all other times. If it rains and there is at least one umbrella with him, he takes it. Otherwise, he gets wet. Let \( X_n \) be the number of umbrellas at his current location after \( n \) trips (so \( n \) even corresponds to home and \( n \) odd to work).

(a) Find the transition probabilities for the Markov chain \( X_n \).

(b) Show that it is reversible and find the stationary distribution.

(c) What is the probability that Tlaloc gets wet on any given trip?

(d) What value of \( p \) maximizes this probability?

(e) What happens if he has \( N \) umbrellas for \( N \) other than 4?

**Solution.**

(a) With states in order 0, 4, 1, 3, 2 we get the transition matrix:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & p & 1-p \\
0 & 0 & p & 1-p & 0 \\
0 & p & 1-p & 0 & 0 \\
p & 1-p & 0 & 0 & 0
\end{pmatrix}
\]

(b) The detailed balance condition gives \( \pi_0 = (1-p)\pi_4 \) and \( \pi_4 = \pi_1 = \pi_3 = \pi_2 \). Thus we can normalize to find

\[
\pi = \left( \frac{1-p}{5-p}, \frac{1}{5-p}, \frac{1}{5-p}, \frac{1}{5-p}, \frac{1}{5-p} \right).
\]

(c) The probability of getting wet is \( p\pi_0 = \frac{p(1-p)}{5-p} \).

(d) Taking derivatives, we find this is maximized at \( p = 5 - \sqrt{20} \approx 0.53 \), and the probability is \( 9 - \sqrt{80} \approx 0.056 \).
(e) In a similar way we can get \( \pi_0 = \frac{1-p}{N+1-p} \) and \( \pi_i = \frac{1}{N+1-p} \) for all other \( i \). Therefore the probability of getting wet is \( \frac{p(1-p)}{N+1-p} \), which is maximized at \( p = N + 1 - \sqrt{N^2 + N} \) with value \( 2N + 1 - 2\sqrt{N^2 + N} \). Since \( \sqrt{N^2 + N} = N + 1/2 - 1/8N + O(N^{-2}) \), the \( p \) that maximizes the probability of getting wet is \( 1/2 + O(1/N) \) and the probability is \( 1/(4N) + O(N^{-2}) \).

Problem 3. Consider the random walk on \( \mathbb{Z} \) with jump probabilities from \( n \) given by \( P_{n,n+1} = p > 1/2 \) and \( P_{n,n-1} = 1 - p \). Find the probability that, starting at 0, the walk returns to 0 at some later time. One method: Let \( q \) be the probability that if we are at \( n \) we ever hit \( n - 1 \). Argue why this does not depend on \( n \). Condition on the first step from \( n \) and find an equation satisfied by \( q \). Use the value of \( q \) to solve the original problem.

Solution. There are several ways to do this. One way: Let \( q \) be as in the problem. If the first step is to \( n - 1 \) then we certainly reach \( n - 1 \). If the first step is to \( n + 1 \), then we must first reach \( n \), and then reach \( n - 1 \). The probability that both of these happen is \( q^2 \), using the MArkov property. Therefore,\[
q = p \cdot q^2 + (1 - p) \cdot 1.
\]
This has two solutions: \( q = 1 \) and \( q = \frac{1-p}{p} \). Since the walk is transient and \( p > 1/2 \), it tends to \( +\infty \), and the probability is not 1, so \( q = \frac{1-p}{p} \).

Now, if we start at 0, and the first step is to \( -1 \), then we will certainly reach 0 at some point. If the first step is to 1, the probability of getting back to 0 is \( q \). Therefore the probability of returning to 0 is \( pq + (1 - p)1 = 2(1 - p) \).

Note: The if \( p < 1/2 \), we get probability \( 2p \).

Another approach: the average step is \( p \cdot 1 + (1 - p)(-1) = 2p - 1 \). Therefore after \( n \) steps the walk is roughly at \( (2p - 1)n \). This means each integer is visited an average of \( \frac{1}{2p-1} \) times. However, if the probability of returning to the starting point is \( \mu \) then the average number of visits is \( \frac{1}{1-\mu} \). This gives \( \frac{1}{1-\mu} = \frac{1}{2p-1} \) so that \( \mu = 2 - 2p \).

Problem 4. Smith has three records A,B,C which he keeps in a stack. (These are like mp3 files, but come in the shape of a physical flat disc!) After he plays a record, he puts it at the top of the stack. His favourite is A, which he selects to listen with probability \( \frac{2}{3} \). He selects B with probability \( \frac{1}{6} \), and he selects C also with probability \( \frac{1}{6} \). This defines a Markov chain, with the state of the system given by the different possible orders for the records. In your solution, order your states in the order: ABC,ACB,BAC,BCA,CAB,CBA.

(a) Write down the transition matrix for the Markov chain.
(b) Determine the stationary distribution of the Markov chain.
(c) Suppose the records are now in order BCA. How many steps will it take, on average, until the books are again in the same order?
Solution. (a) The transition matrix is

\[
\begin{pmatrix}
ABC & ACB & BAC & BCA & CAB & CBA \\
2/3 & 0 & 1/6 & 0 & 1/6 & 0 \\
0 & 2/3 & 1/6 & 0 & 1/6 & 0 \\
2/3 & 0 & 1/6 & 0 & 0 & 1/6 \\
2/3 & 0 & 0 & 1/6 & 0 & 1/6 \\
0 & 2/3 & 0 & 1/6 & 1/6 & 0 \\
0 & 2/3 & 0 & 1/6 & 0 & 1/6
\end{pmatrix}
\]

(b) Using $\pi P = \pi$ and $\sum \pi_x = 1$ we can solve to find

\[
\pi_{ABC} = \pi_{ACB} = \frac{1}{3}, \quad \pi_{BAC} = \pi_{CAB} = \frac{2}{15}, \quad \pi_{BCA} = \pi_{CBA} = \frac{1}{30}.
\]

This can also be found by hand: using the symmetry between B and C we get that the three pairs of values are equal. The probability that the last record picked is A is $2/3$, so $\pi_{ABC} = \pi_{ACB} = \frac{1}{3}$. The probability that the last record picked is B is $1/6$, so $\pi_{BAC} + \pi_{BCA} = \frac{1}{6}$. Using one row from $\pi P = \pi$ gives an extra equation to find the values.

(c) The average time from $x$ to return to $x$ is $1/\pi_x$, which in this case is 30.

Problem 5. Simulate $10^6$ steps of a random walk on a $1000 \times 1000$ grid. For each vertex, count the number of times it is visited. Submit a histogram of the result, as well as a density plot of the result. Repeat the experiment with $10^7$ steps.

Note: if the random walk is at the edge of the grid, prevent it from leaving.

Solution. Plots of the number of visits: Plots of the trajectories:

In the first figure, I excluded the number of unvisited sites, which is almost all. The most visited site up to time $n$ is visited roughly $\log n$ times in the infinite grid, which is why the maximal value with $10^7$ steps is not much more than with $10^6$ steps. If we run the process for a very long time, eventually each vertex is visited roughly $n/10^6$ times.

Sample code:
from random import *
import numpy as np
import matplotlib.pyplot as plt

# the following uses periodic boundary conditions.

def probab(p):
    return random() < p

def RW2d(N=1000000, L=1000):
    A = np.zeros((L,L), dtype=int)
    x = y = L//2
    for i in range(N):
        A[x, y] += 1
        d = 1 if probab(1/2) else -1
        if probab(1/2): x = (x+d)%L
        else: y = (y+d)%L
    return A

def get_hist(A):
    m = max(A.flatten())
    L = len(A)
    H = np.zeros(m+1, dtype=int)
    for x in range(L):
        for y in range(L):
            H[A[x, y]] += 1
    return H

A6 = RW2d(1000000)
H = get_hist(A6)
plt.figure()
plt.plot(H[1:])
plt.savefig('hw11_q5a.pdf')
plt.figure()
plt.imshow((A6>0) + A6/20,cmap='Greys')
plt.savefig('hw11_q5c.pdf')

A7 = RW2d(10000000)
H = get_hist(A7)
plt.figure()
plt.plot(H)
plt.savefig('hw11_q5b.pdf')
plt.figure()
plt.imshow((A7>0) + A7/20,cmap='Greys')
plt.savefig('hw11_q5d.pdf')
With thanks to Randall Munroe (XKCD)\(^1\).

\(^1\)note: python 3 requires \texttt{print("Hello, world!")}