Problem 1. Find the stationary distribution (if it exists) of the following birth and death chains, with states are \( \{0,1,\ldots\} \), and in all three, \( P_{0,-1} = 0 \) and \( P_{01} = 1 \).

(a) \( P_{i,i+1} = \frac{1}{i+1} \) and \( P_{i,i-1} = \frac{i}{i+1} \).

(b) \( P_{i,i+1} = \frac{i}{i+1} \) and \( P_{i,i-1} = \frac{1}{i+1} \).

Problem 2. Tlaloc has 4 umbrellas, each either at home or at work. Each time he goes to work or back, it rains with probability \( p \), independently of all other times. If it rains and there is at least one umbrella with him, he takes it. Otherwise, he gets wet. Let \( X_n \) be the number of umbrellas at his current location after \( n \) trips (so \( n \) even corresponds to home and \( n \) odd to work).

(a) Find the transition probabilities for the Markov chain \( X_n \).

(b) Show that it is reversible and find the stationary distribution.

(c) What is the probability that Tlaloc gets wet on any given trip?

(d) What value of \( p \) maximizes this probability?

(e) What happens if he has \( N \) umbrellas for \( N \) other than 4?

Problem 3. Consider the random walk on \( \mathbb{Z} \) with jump probabilities from \( n \) given by \( P_{n,n+1} = p > 1/2 \) and \( P_{n,n-1} = 1-p \). Find the probability that, starting at 0, the walk returns to 0 at some later time. One method: Let \( q \) be the probability that if we are at \( n \) we ever hit \( n-1 \). Argue why this does not depend on \( n \). Condition on the first step from \( n \) and find an equation satisfied by \( q \). Use the value of \( q \) to solve the original problem.

Problem 4. Smith has three records A,B,C which he keeps in a stack. (These are like mp3 files, but come in the shape of a physical flat disc!) After he plays a record, he puts it at the top of the stack. His favourite is A, which he selects to listen with probability \( 2/3 \). He selects B with probability \( 1/6 \), and he selects C also with probability \( 1/6 \). This defines a Markov chain, with the state of the system given by the different possible orders for the records. In your solution, order your states in the order: ABC,ACB,BAC,BCA,CAB,CBA.

(a) Write down the transition matrix for the Markov chain.

(b) Determine the stationary distribution of the Markov chain.

(c) Suppose the records are now in order BCA. How many steps will it take, on average, until the books are again in the same order?

Problem 5. Simulate \( 10^6 \) steps of a random walk on a \( 1000 \times 1000 \) grid. For each vertex, count the number of times it is visited. Submit a histogram of the result, as well as a density plot of the result. Repeat the experiment with \( 10^7 \) steps.

Note: if the random walk is at the edge of the grid, prevent it from leaving.

Additional problems:

(a) Write down a few \( 6 \times 6 \) stochastic matrix and determine its irreducible classes, recurrence and periodicity of states.

(b) Write down a 4 state irreducible Markov chain with \( P_{ii} = 0 \) for all \( i \) and find its stationary distribution.
(c) A matrix is doubly stochastic if it is stochastic and every columns also adds up to 1. Show that if $P$ is doubly stochastic then the stationary distribution is $\pi_i = 1/n$ where $n$ is the number of states.

(d) Find the stationary distribution of the following Markov chain. The states are 0, 1, 2, \ldots, N. From each $i$ except 0 and $N$ the walk moves with equal probabilities to $i \pm 1$. From 0 the walk moves always to 1. From $N$ the walk moves with equal probabilities to 0 and $N - 1$. 