Problem 1. In each of (a)–(d), determine whether or not the given Markov chain is irreducible, and identify each state as recurrent or transient, and as periodic or aperiodic. In (a) and (b), the state space is \( S = \{1, \ldots, 5\} \).

(a) \[
P = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

(b) \[
P = \begin{pmatrix}
0 & \frac{4}{5} & \frac{1}{5} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(c) The simple symmetric random walk on \( \mathbb{Z}^d \) (answer for each \( d \geq 1 \)).

(d) The random walk on \( \mathbb{Z} \) with probability \( \frac{1}{3} \) of moving right and \( \frac{2}{3} \) of moving left.

Problem 2. Consider the Markov chain with state space \( S = \{1, 2, 3\} \) and transition matrix

\[
P = \begin{pmatrix}
\frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\
0 & \frac{1}{3} & \frac{2}{3} \\
\frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix}
\]

Suppose \( P(X_0 = 1) = P(X_0 = 2) = \frac{1}{2} \). What is \( P(X_3 = 3) \)?

Problem 3. Chapter 4, problem 7 (concerning the weather model of Example 4.4).

Problem 4. There are \( n \) coins on the table. Each step we choose at random one of the coins and toss it again. Let \( X_m \) be the number of heads showing. Show that this chain has transition probabilities

\[
P_{ii} = \frac{1}{2}, \quad P_{i,i-1} = \frac{i}{2n}, \quad P_{i,i+1} = \frac{n-i}{2n}.
\]

Problem 5. (a) For the gambler’s ruin problem, let \( M_k \) denote the expected number of games that will be played when Mark initially has \$k, and stops at 0 or \( n \). Let \( p \) be the probability of winning each bet, and \( q = 1 - pp \). Show that \( M_0 = M_n = 0 \) and

\[
M_k = 1 + pM_{k+1} + qM_{k-1}
\]

for \( 0 < k < n \). (Hint: Compute the expectation of the number of games \( X \) by conditioning on the outcome of the first game. If \( A \) is the event that Mark wins the first game,

\[
\]

(b) Solve the equations in (a) to obtain

\[
M_k = k(n - k) \text{ if } p = 1/2,
\]

and

\[
M_k = \frac{k}{q - p} - \frac{n}{q - p} \frac{1 - \alpha^k}{1 - \alpha^n} \text{ if } p = 1/2,
\]
where $\alpha = q/p$. To do this, proceed as follows. First, find the general solution to the homogeneous equation $M_k = pM_{k+1} + qM_{k-1}$ (as done in class). Next, find a particular solution to the inhomogeneous equation $M_k = 1 + pM_{k+1} + qM_{k-1}$ (try $M_k = ck^2$ for $p = 1/2$ and $M_k = ck$ for $p \neq 1/2$). Add the general solution of the homogeneous equation to the particular solution of the inhomogeneous equation. Finally, solve for the two unknown constants in the general solution by using the boundary conditions.

**Problem 6.** Simulate the previous process with 1000 coins. Start with 0 heads, and run the process for 20000 steps. Keep track of how many times each state is visited, from time 0 to time 1000, from 1000–2000, and finally, in times 10000-20000. Submit your code and histograms of those.

Extra practice problems (do not hand in)
(a) Chapter 4: 2,13,14,15,38,57
(b) Write down a 6 × 6 stochastic matrix and determine its irreducible classes, recurrence and periodicity of states.
(c) Write down a 4 state irreducible markov chain with $P_{ii} = 0$ for all $i$ and find its stationary distribution.
(d) A matrix is doubly stochastic if it is stochastic and every columns also adds up to 1. Show that if $P$ is doubly stochastic then the stationary distribution is $\pi_i = 1/n$ where $n$ is the number of states.
(e) Find the stationary distribution of the following Markov chain. The states are 0, 1, 2, ..., $N$. From each $i$ except 0 and $N$ the walk moves with equal probabilities to $i \pm 1$. From 0 the walk moves always to 1. From $N$ the walk moves with equal probabilities to 0 and $N - 1$. 