*This midterm has **4 questions** on **6 pages**, for a total of 35 points.*

Duration: 50 minutes

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- Don't think about pink elephants.
- Does anyone read these things?

Full Name (including all middle names): ____________________

Student-No: ____________________

Signature: ____________________

Question:	1	2	3	4	Total
Points:	15	6	6	8	35
Score:					

SHORT ANSWER QUESTIONS.

Please show your work and also underline your answer.

Each question is worth 3 marks, but an incorrect answer will be given at most 1 mark.

Unless otherwise stated, it is not necessary to simplify your answers.

3 marks 1. (a) Let $y = \tan(\arccos(x))$. What is $\frac{\mathrm{d}y}{\mathrm{d}x}$? (remember arccos is inverse-cosine).

Solution:

$$\begin{aligned} y &= \tan(\arccos(x)) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \sec^2(\arccos x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) \\ &= \frac{1}{\cos^2(\arccos x)} \cdot \frac{-1}{\sqrt{1-x^2}} \\ &= \frac{1}{x^2} \cdot \frac{-1}{\sqrt{1-x^2}} = \frac{-1}{x^2\sqrt{1-x^2}} \end{aligned}$$

3 marks (b) What is $\lim_{x\to+\infty} \frac{x+2}{\sqrt{4x^2+x}}$?

Solution:

$$\begin{aligned} \lim_{x\to\infty} \frac{x+2}{\sqrt{4x^2+x}} &= \lim_{x\to\infty} \frac{x+2}{\sqrt{4x^2+x}} \cdot \frac{1/x}{1/x} \\ &= \lim_{x\to\infty} \frac{1+2/x}{\sqrt{4+1/x}} \\ &= \frac{1}{\sqrt{4}} = 1/2 \end{aligned}$$

3 marks (c) Use a linear approximation to estimate $\sqrt{3.9}$.

Solution: Let $f(x) = \sqrt{x}$ and we expand around $a = 4$. We need the first derivative — $f'(x) = \frac{1}{2\sqrt{x}}$. The linear approximation is

$$\begin{aligned} L(x) &= f(4) + f'(4)(x-4) \\ &= 2 + \frac{1}{4}(x-4) \\ L(4.1) &= 2 + \frac{1}{4} \cdot -0.1 \\ &= 2 - 0.025 = 1.975 \end{aligned}$$

3 marks

(d) Find the derivative of the following function

$$g(t) = \frac{(1+t)^{3/2}(2-\cos(t))^7}{(1+e^t)^{2/5}}.$$

Solution: Use log-differentiation

$$\ln g = \frac{3}{2}\ln(1+t) + 7\ln(2-\cos t) - \frac{2}{5}\ln(1+e^t)$$

differentiate wrt t

$$\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{3}{2(1+t)} + \frac{7\sin t}{2-\cos(t)} - \frac{2e^t}{5(1+e^t)}$$

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \left(\frac{3}{2(1+t)} + \frac{7\sin t}{2-\cos(t)} - \frac{2e^t}{5(1+e^t)}\right)\frac{(1+t)^{3/2}(2-\cos(t))^7}{(1+e^t)^{2/5}}$$

3 marks

(e) If $x^3 - y^3 = e^y$ what is $\frac{\mathrm{d}y}{\mathrm{d}x}$?

Solution:

$$x^3 - y^2 = e^y$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^y\right)$$

$$3x^2 - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = e^y\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$3x^2 = (2y + e^y)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{2y + e^y}$$

FULL-SOLUTION PROBLEMS

In questions 2–4, justify your answers and **show all your work.** If you need more space, use the back of the *previous* page.

6 marks

2. A 1kg lump of unknown material is decaying radioactively. After 20 minutes 100g has decayed.

(a) What is the half-life of the material?

Solution:

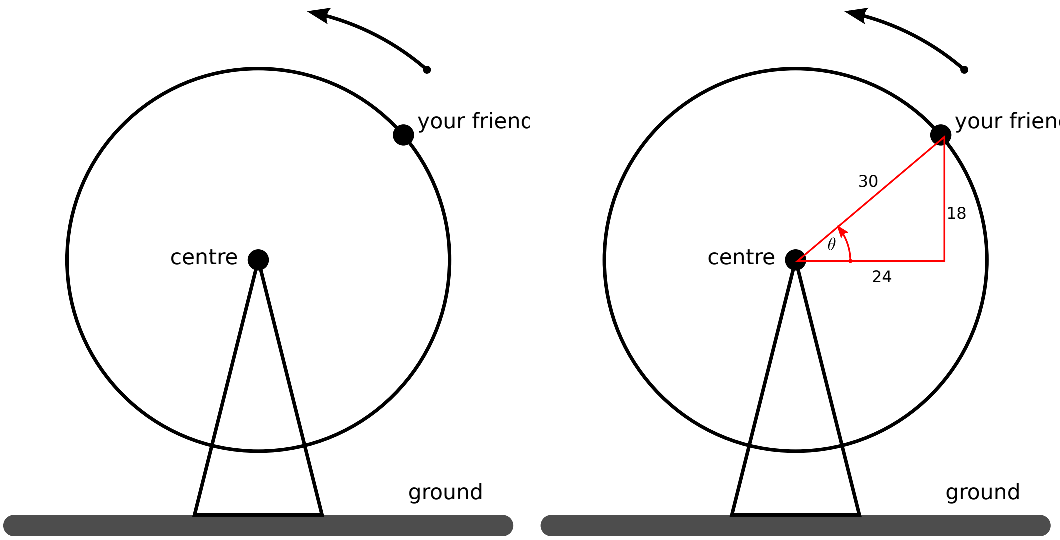
$$\begin{aligned} P(t) &= P(0)e^{kt} & P(0) &= 1kg \\ P(20) = 0.9 &= e^{20k} & k &= \ln(0.9)/20 \\ 0.5 &= e^{t\ln(0.9)/20} \\ \ln(0.5) &= t\ln(0.9)/20 \\ t &= 20\ln(0.5)/\ln(0.9) \text{ minutes} \end{aligned}$$

(b) How long until only 250g remain?

Solution: In one "half-life" there will be 500g remaining. In two "half-lives" there will be 250g remaining. Hence $40\ln(0.5)/\ln(0.9)$ minutes.

6 marks

3. Your friend is riding a big circular Ferris wheel with radius 30m. It completes one rotation every 5 minutes. How fast is your friend rising when they are 18m higher than the centre of the wheel? Include units in your answer.



Solution:

- Since the wheel rotates once in 5 minutes, $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 2\pi/5$ radians per minute.
- The height above the centre of the wheel is given by $h = 30 \sin\theta$.
- Thus $\frac{\mathrm{d}h}{\mathrm{d}t} = 30\cos\theta \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$.
- When they are at $h = 18$, $\sin\theta = 18/30 = 3/5$. Since $\sin^2\theta + \cos^2\theta = 1$, $\cos\theta = 4/5$. **It really helps to know $3^2 + 4^2 = 5^2$ — you should!**
- Putting this together gives $\frac{\mathrm{d}h}{\mathrm{d}t} = 30 \cdot \frac{4}{5} \cdot \frac{2\pi}{5} = \frac{48\pi}{5}$ metres per minute.

8 marks 4. Consider the function $f(x) = e^x \sin(x)$.

(a) Write down the 3rd degree Maclaurin polynomial for f and so approximate $f(1/2)$.

Solution:

$$\begin{aligned} f(x) &= e^x \sin x & f(0) &= 0 \\ f'(x) &= e^x(\sin x + \cos x) & f'(0) &= 1 \\ f''(x) &= e^x(\sin x + \cos x + \cos x - \sin x) & & \\ &= 2e^x \cos x & f''(0) &= 2 \\ f'''(x) &= 2e^x(\cos x - \sin x) & f'''(0) &= 2 \\ T_3(x) &= x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 & & \\ &= x + x^2 + \frac{1}{3}x^3 & & \end{aligned}$$

Thus

$$T_3(1/2) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \times \frac{1}{3} = \frac{12+6+1}{24} = \frac{19}{24}$$

(b) Estimate the error in this approximation.

Solution: We need $f^{(4)}(x)$

$$f^{(4)}(x) = 2e^x(\cos x - \sin x) + 2e^x(-\sin x - \cos x) = -4e^x \sin x$$

Plugging this into the remainder formula

$$\begin{aligned} R_4(x) &= \frac{1}{4!} f^{(4)}(c)(x-a)^4 & \text{with } 0 < c < 1/2 \\ |R_4(1/2)| &= \frac{1}{24}|-4e^c \sin c|\frac{1}{16} & |\sin c| < 1 \\ &\le \frac{4}{24} \cdot \frac{1}{16} \cdot |e^c| & e^c < 3 \\ &\le \frac{12}{24} \cdot \frac{1}{16} = \frac{1}{32} & \end{aligned}$$