*This midterm has **4 questions** on **5 pages**, for a total of 35 points.*

Duration: 50 minutes

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations, answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Full Name (including all middle names): ____________________

Student-No: ____________________

Signature: ____________________

Question:	1	2	3	4	Total
Points:	15	6	7	7	35
Score:					

SHORT ANSWER QUESTIONS.

Please show your work and also underline your answer.

Each question is worth 3 marks, but an incorrect answer will be given at most 1 mark.

Unless otherwise stated, it is not necessary to simplify your answers.

3 marks 1. (a) Evaluate the limit $\lim_{x\to 2} \frac{x-2}{x^2-4}$ or determine that it does not exist.

Solution: At $x = 2$ both numerator and denominator are zero, so we first simplify.

$$\begin{aligned}
\lim_{x\to 2} \frac{x-2}{x^2-4} &= \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} \\
&= \lim_{x\to 2} \frac{1}{(x+2)} \\
&= \frac{\lim_{x\to 2} 1}{\lim_{x\to 2}(x+2)} = \frac{1}{4}
\end{aligned}$$

3 marks (b) Evaluate the limit $\lim_{t\to 0} \frac{\sqrt{t+9}-3}{t}$ or determine that it does not exist.

Solution: At $t = 0$ both numerator and denominator are zero, so we need to simplify.

$$\begin{aligned}
\lim_{t\to 0} \frac{\sqrt{t+9}-3}{t} &= \lim_{t\to 0} \frac{\sqrt{t+9}-3}{t} \cdot \frac{\sqrt{t+9}+3}{\sqrt{t+9}+3} \\
&= \lim_{t\to 0} \frac{(t+9)-9}{t(\sqrt{t+9}+3)} \\
&= \lim_{t\to 0} \frac{1}{\sqrt{t+9}+3} = \frac{1}{6}
\end{aligned}$$

3 marks (c) What value of c makes the following function continuous?

$$h(x) = \begin{cases} 3x+2 & \text{if } x < c \\ 4-x & \text{if } x \geq c \end{cases}$$

Solution: We need the left-limit to equal the right limit.

$$\lim_{x\to c^-} h(x) = 3c+2$$
$$\lim_{x\to c^+} h(x) = 4-c$$

So we need $3c+2 = 4-c$ or $4c = 2$. Hence $c = 1/2$.

3 marks

(d) Find the derivative of $f(x) = \frac{e^x}{x^3+3}$

Solution: We use the quotient rule

$$\begin{aligned} f'(x) &= \frac{(x^3+3)\frac{\mathrm{d}}{\mathrm{d}x}(e^x) - e^x\frac{\mathrm{d}}{\mathrm{d}x}(x^3+3)}{(x^3+3)^2} \\ &= \frac{(x^3+3)e^x - (3x^2+0)e^x}{(x^3+3)^2} \\ &= \frac{(x^3-3x^2+3)e^x}{(x^3+3)^2} \end{aligned}$$

3 marks

(e) Find the *second* derivative of $f(x) = x^2e^x$.

Solution:

- We apply the product rule to get

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right)e^x + \frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right)x^2 = (2x+x^2)e^x$$

- We apply it a second time to get

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(2x+x^2\right)e^x + \frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right)\left(2x+x^2\right) \\ &= (2+2x)e^x + (2x+x^2)e^x = (2+4x+x^2)e^x \end{aligned}$$

FULL-SOLUTION PROBLEMS

In questions 2–4, justify your answers and **show all your work.** If you need more space, use the back of the *previous* page.

6 marks 2. Let

$$f(x) = e^{\sin x} - x$$

Explain why $f(x)$ is continuous for all x. Then use the Intermediate Value Theorem to prove there is a point c in the open interval $(0, \pi)$ so that $f(c) = 0$.

Solution:

- Since $\sin x, e^x$ and x are continuous on all x. Since sums and compositions of continuous functions are continuous it follows that $f(x)$ is continuous everywhere (and in particular on $[0, \pi]$). Hence we can use the IVT.
- When $x = 0$, $f(x) = e^{\sin 0} - 0 = e^0 = 1$.
- When $x = \pi$, $f(x) = e^{\sin \pi} - \pi = e^0 - \pi = 1 - \pi < 0$.
- Since $f(0) > 0$ and $f(\pi) < 0$, by the IVT, there is some c between 0 and π so that $f(c) = 0$.

7 marks 3. Find the equation of a line that is tangent to the curve $f(x) = x^2 - 2x$ and passes through the point $(2, -4)$.

Solution:

- Assume the line intersects the curve at $x = a$.
- Then the gradient of the line is $\frac{a^2-2a+4}{a-2}$
- This must be equal to the derivative of the curve $2a - 2$. So

$$\begin{aligned} \frac{a^2 - 2a + 4}{a - 2} &= 2(a - 1) \\ a^2 - 2a + 4 &= 2(a - 1)(a - 2) = 2a^2 - 6a + 4 \\ 0 &= a^2 - 4a = a(a - 4) \end{aligned}$$

- Thus $a = 0, 4$.
- If $a = 0$ then $m = f'(0) = -2$. Hence $c = 0$ and so $y = -2x$.
- If $a = 4$ then $m = f'(4) = 6$. Hence $c = -16$ and so $y = 6x - 16$.

7 marks 4. Let $g(t) = \dfrac{t}{2+t}$. Use the definition of the derivative to find $\frac{\mathrm{d}g}{\mathrm{d}t}$.

You must show your work.

No credit will be given on this problem for using derivative formulas.

Solution:

$$\begin{aligned}
\frac{\mathrm{d}g}{\mathrm{d}t} &= \lim_{h\to 0} \frac{g(t+h)-g(t)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{t+h}{2+t+h} - \dfrac{t}{2+t}}{h} \\
&= \lim_{h\to 0} \cdot \frac{(t+h)(2+t) - t(2+t+h)}{h(2+t+h)(2+t)} \\
&= \lim_{h\to 0} \cdot \frac{(2t+2h+t^2+th) - (2t+t^2+th)}{h(2+t+h)(2+t)} \\
&= \lim_{h\to 0} \frac{2h}{h(2+t+h)(2+t)} \\
&= \lim_{h\to 0} \frac{2}{(2+t+h)(2+t)} = \frac{2}{(2+t)^2}
\end{aligned}$$