Introduction to étale cohomology

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Outline

1. Grothendieck topologies
2. Presheaves and sheaves on sites
3. Cohomology of sheaves
4. Étale morphisms and sites
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The site of a topological space

Let

- \( X \) be a topological space,
- \( X_{cl} \) be the set of all open subsets of \( X \),
- \( \text{cov}(X_{cl}) \) be the set of families \( \{U_i \to U\} \) which are coverings of an \( U \subseteq X \) open.

\( X_{cl} \) becomes a category if we set:

\[
\text{Hom}(U, V) = \begin{cases} 
\emptyset & \text{if } U \not\subseteq V \\
\text{inclusion } U \to V & \text{if } U \subseteq V.
\end{cases}
\]

In this category if \( U_1 \to U \) and \( U_2 \to U \) are arrows, then their fiber product is their intersection:

\[
U_1 \times_U U_2 = U_1 \cap U_2.
\]
Properties of $\text{cov}(X_{cl})$

Proposition

(T1) For $U_i \to U \in \text{cov}(X_{cl})$ and a morphism $V \to U$ in $X_{cl}$ all fibre products $U_i \times_U V$ exist and $\{U_i \times_U V \to V\} \in \text{cov}(X_{cl})$.

(T2) Given $\{U_i \to U\} \in \text{cov}(X_{cl})$ and a family $\{V_{ij} \to U_i\} \in \text{cov}(X_{cl})$ for all $i \in I$, the family $\{V_{ij} \to U\}$ obtained by composition of morphisms, also belongs to $\text{cov}(X_{cl})$.

(T3) If $V \to U$ is an isomorphism in $X_{cl}$, then $\{V \to U\} \in \text{cov}(X_{cl})$.

In fact, the set $\text{cov}(X_{cl})$ describes the topology of $X$. 
Grothendieck topologies

Definition
A topology (or site) $T$ consists of a category $\text{cat}(T)$ and a set $\text{cov}(T)$ of coverings, i.e. families $\{U_i \to U\}_{i \in I}$ of morphisms in $\text{cat}(T)$, which satisfy (T1), (T2) and (T3).

Definition
A morphism $f : T \to T'$ of topologies is a functor $f : \text{cat}(T) \to \text{cat}(T')$ of the underlying categories with the following two properties

(a) $\{U_i \to U\} \in \text{cov}(T) \Rightarrow \{f(U_i) \to f(U)\} \in \text{cov}(T')$

(b) For $\{U_i \to U\} \in \text{cov}(T)$ and a morphism $V \to U$ in $\text{cat}(T)$ the canonical morphism

$$f(U_i \times_U V) \to f(U_i) \times_{f(U)} f(V)$$

is an isomorphism for all $i$. 
Presheaves and sheaves on topological spaces

Let $\mathcal{C}$ be a category (e.g. $\text{Sets}$ or $\text{Ab}$).
If $X$ is a topological space, a presheaf on $X$ with values in $\mathcal{C}$ is a functor

$$F : X^\text{op} \to \mathcal{C}.$$  

For every presheaf $F$ of sets on $X$ and every $\{U_i \to U\} \in \text{cov}(T)$ there is a diagram

$$F(U) \to \prod_i F(U_i) \xrightarrow{\text{pr}_1^*} \prod_{i,j} F(U_i \times_U U_j).$$

Here $F(U) \to \prod_i F(U_i)$ is induced by the restrictions

$F(U) \to F(U_i)$, and $\prod_i F(U_i) \xrightarrow{\text{pr}_1^*} \prod_{i,j} F(U_i \times_U U_j)$ is induced by

$\text{pr}_1^* : F(U_i) \xrightarrow{\text{pr}_1^*} \prod_j F(U_i \times_U U_j)$ for each $i$ ($\text{pr}_2^*$ similarly).
The sheaf condition

The presheaf $F : X^{\text{op}}_{cl} \to C$ is a sheaf, if the following holds:

**\( (SH) \)** For every $\{U_i \to U\} \in \text{cov}(T)$ and every $a_i \in F(U_i)$, such that $\text{pr}_1^*(a_i) = \text{pr}_2^*(a_j) \in F(U_i \times_U U_j) (= F(U_i \cap U_j))$ for every $i, j$, there is a unique $a \in F(U)$ whose pullback to $F(U_i)$ is $a_i$.

Equivalently:

**\( (SH') \)** For every $\{U_i \to U\} \in \text{cov}(T)$ the diagram

\[ F(U) \to \prod_i F(U_i) \xrightarrow{\text{pr}_1^*} \prod_i F(U_i \times_U U_j) \]

has the properties:

- $F(U) \to \prod_i F(U_i)$ is injective,
- $\text{Im}(F(U) \to \prod_i F(U_i)) = \{(a_i) \in \prod_i F(U_i) | \text{pr}_1^*(a_i) = \text{pr}_2^*(a_j) \forall i, j\}$. 
Presheaves and sheaves on sites

Let $\mathcal{C}$ be an category and $\mathcal{T}$ a topology.

**Definition**

1. A *presheaf* on $X$ with values in $\mathcal{C}$ is a contravariant functor

   $$F : \mathcal{T} \to \mathcal{C},$$

2. $F$ is a *sheaf* if it moreover satisfies (SH), or equivalently (SH’).

3. A *morphism of (pre)sheaves* $F \to G$ is a natural transformation of functors.

Abelian presheaves and sheaves on a topology $\mathcal{T}$ form abelian categories $\mathcal{P}$ and $\mathcal{S}$. 
All sheaves are presheaves, so there is an inclusion functor

\[ i : S \to \mathcal{P} . \]

**Theorem**

*There exist a left-adjoint functor \( \# : S \to \mathcal{P} \) of \( i \).*

**Definition**

For each \( F \in \mathcal{P} \), the sheaf \( F\# \) is called the *sheaf associated to the presheaf \( F \).*

This is a universal construction in the sense, that each morphism from \( F \) to an abelian sheaf \( G \) factors uniquely as \( F \to F\# \to G \).
Refinement of coverings

Definition
\[ \{U_j \rightarrow U\}_{j \in J} \rightarrow \{U_i \rightarrow U\}_{i \in I} \] if there is an \( \varepsilon : J \rightarrow I \), such that
\( \{U_j' \rightarrow U\} \) factorizes as

\[ U_j' \xrightarrow{} U \xrightarrow{} U \]

\[ \triangleleft \]

\[ U_{\varepsilon(j)} \]

\( \sim \) an inverse system of covers can be constructed.
Reminder on derived functors

- An abelian category $\mathcal{C}$ has enough injectives, if for each object $A$ there is a monomorphism $A \to I$ into an injective object of $\mathcal{C}$.
- If $F : \mathcal{C} \to Ab$ is an additive, left-exact functor, then its derived functor is defined as
  1. Construct an injective resolution of $X$:
     \[ 0 \to X \to I^0 \to I^1 \to I^2 \ldots . \]
  2. Apply $F$ on it and chop off the first term:
     \[ 0 \to F(I^0) \to F(I^1) \to F(I^2) \ldots . \]
  3. The $i$-th derived functor of $F$ on $X$ is
     \[ R^i F(X) := \ker(d^i)/\im(d^{i-1}) . \]
Cohomology of sheaves

$S$ has enough injectives $\Rightarrow$ we can take right derived functors. Consider for a fixed $U \in \mathcal{T}$ the section functor

$$\Gamma_U : S \to \text{Ab},$$

defined by $\Gamma_U(F) = F(U)$. This is additive, left-exact, and

\[
\begin{array}{ccc}
S & \xrightarrow{\Gamma_U} & \text{Ab} \\
\downarrow i & & \downarrow \Gamma_U \\
\mathcal{P} & \xleftarrow{i} & \\
\end{array}
\]

Definition

For $q \geq 0$, the $q$-th cohomology group of $U$ with values in $F$ is

$$H^q(U, F) := R^q\Gamma_U(F).$$
Direct/inverse images for presheaves

Let \( f : T \to T' \) be a morphism of topologies, and \( \mathcal{P}, \mathcal{S} \) and \( \mathcal{P}', \mathcal{S}' \) be the categories of abelian (pre)sheaves on \( T \) and \( T' \), respectively.

**Definition**

If \( F' \) is an abelian presheaf on \( T' \), then its direct image \( f^p F' \) is the presheaf on \( T \) given by

\[
U \mapsto f^p F'(U) = F'(f(U)),
\]

for \( U \in T \). This is functorial in \( F' \Rightarrow \) we get an additive, exact functor:

\[
f' : T' \to T.
\]

**Theorem**

The functor \( f^p \) has a left adjoint \( f_p \), which is right-exact.
These induce functors between $S$ and $S'$ as well:

1. 
   \[ f^s : S' \to S, \quad f^s = \# \circ f^p \circ i', \]

2. 
   \[ f_s : S \to S', \quad f_s = \#' \circ f_p \circ i. \]
Cohomology and limits

Definition
A topology $T$ is noetherian, if each object of $T$ is quasi-compact.

Theorem
Assume $T$ is noetherian, and $I$ is a category with a sensible definition of limit (pseudofiltered category). Then

$$\lim_{\overset{\longrightarrow}{I}} H^q(U, F_i) \simeq H^q(U, \lim_{\overset{\longrightarrow}{I}} F_i)$$
The implicit function theorem

Theorem

If $f_1, \ldots, f_k$ are analytic functions around $x \in \mathbb{C}^{k+n}$, such that
\[ \det_{1 \leq i, j \leq k} \left( \frac{\partial f_i}{\partial x_j} \right)(x) \neq 0, \]
then the projection

\[ (f_1 = \cdots = f_k = 0) \rightarrow \mathbb{C}^n \]

\[ (x_1, \ldots, x_{k+n}) \mapsto (x_{k+1}, \ldots, x_{k+n}) \]

is a local analytic isomorphism around $x$. 
This is not true in the Zariski topology of AG

Example

\[ V(x_1^2 - x_2) \to \mathbb{A}^1, \quad (x_1, x_2) \mapsto x_2. \]

At \( x = (1, 1) \) the conditions of IFT are satisfied:

\[
\frac{\partial}{\partial x_1}(x_1^2 - x_2) \bigg|_{x=2} = 2x_1 \bigg|_{x=2} = 2 \neq 0.
\]

But for all \( U \subset V(x_1^2 - x_2) \) Zariski open containing \( x \) the projection to \( x_2 \) is not even a bijection: except for finitely many values of \( a \), \((+\sqrt{a}, a), (-\sqrt{a}, a) \in U \Rightarrow a \) has two preimages.
Étale morphisms

Definition

1. The morphism
   \[ X = \text{Spec} R[ x_1, \ldots, x_n ]/( f_1, \ldots, f_k ) \to \text{Spec} R = Y \] is étale in \( x \in X \), if
   \[ \det_{1 \leq i, j \leq k} \left( \frac{\partial f_i}{\partial x_j} \right) (x) \neq 0. \]

2. The finite type morphism \( f : X \to Y \) is étale, if for all \( x \in X \) there are open neighbourhoods \( x \in U \subset X \) and \( f(x) \in V \subset Y \) such that \( F(U) \subset V \) and \( f|_U \) is étale:

\[
\begin{array}{ccc}
U & \leftarrow & \text{Spec} R[ x_1, \ldots, x_n ]/( f_1, \ldots, f_n ) \\
\downarrow f|_U & & \downarrow \\
V & \leftarrow & \text{Spec} R
\end{array}
\]
The étale site of a scheme

Idea: we change the topology in order for the IFT to hold. We require that open subsets are given by étale morphisms. \(\Rightarrow\) we need a Grothendieck topology!

Definition

- \(Et/X = \text{category of étale } X\text{-schemes}\)
  - \(\text{ob}(Et/X) = \{ Y \to X \text{ étale} \}\)
  - \(\text{Hom}(Y_1 \to X, Y_2 \to X) = \)

\[
\begin{array}{c}
Y_1 \rightarrow Y_2 \\
\downarrow \quad \downarrow \\
X && \text{commutative}
\end{array}
\]
The étale site of a scheme

Definition

- A family \( \{ X_i' \xrightarrow{\varphi_i} X' \} \) of morphisms in \( Et/X \) is called surjective if \( X' = \bigcup_i \varphi_i(X_i') \)
- The étale site \( X_{\text{ét}} \) of \( X \):
  - \( \text{cat}(X_{\text{ét}}) = Et/X \)
  - \( \text{cov}(X_{\text{ét}}) = \) set of surjective families of morphisms in \( Et/X \)
  - Remark: these satisfy the axioms T1, T2 and T3.
- \( \tilde{X}_{\text{ét}} = \) category of abelian sheaves on \( X_{\text{ét}} \).
Zariski and étale cohomology

Proposition

Open immersions are étale.

Corollary

1. Let $X_{\text{Zar}}$ be the topology of open sets of the scheme $X$. Then the inclusion

$$\varepsilon : X_{\text{Zar}} \to X_{\text{ét}}$$

is a morphism of topologies.

2. By spectral sequence arguments there is a functorial morphism

$$H^p_{\text{Zar}}(X, R^q \varepsilon^s(F)) \to H^{p+q}_{\text{ét}}(X, F),$$

which is in general not an isomorphism.
Equivalent conditions of étaleness

Theorem
For a morphism of schemes $f : X \to Y$ the followings are equivalent:

1. $f$ is étale
2. $f$ is smooth and unramified
3. $f$ is smooth and of relative dimension 0
4. $f$ is flat, locally of finite presentation, and for every $y \in Y$, the fiber $f^{-1}(y)$ is a disjoint union of points, each of which is a finite separable field extension of the residue field $\kappa(y)$.

Proposition
Étale morphisms are preserved under composition and base change.
Cohomology of curves

Theorem

$X$, smooth projective algebraic curve over $\mathbb{C}$ with genus $g$. Then

$$H^0(X_{\text{an}}, \mathbb{Z}) = \mathbb{Z},$$

$$H^1(X_{\text{an}}, \mathbb{Z}) = \mathbb{Z}^{2g},$$

$$H^2(X_{\text{an}}, \mathbb{Z}) = \mathbb{Z}.$$

Theorem

$X$, smooth projective algebraic curve over $k$ (algebraically closed) with genus $g$. $(\text{char } k, n) = 1$. Then

$$H^0(X_{\text{ét}}, \mu_n) = \mu_n(k),$$

$$H^1(X_{\text{ét}}, \mu_n) = (\mu_n(k))^{2g},$$

$$H^2(X_{\text{ét}}, \mu_n) = \mu_n(k).$$
Cohomology of fields

Let $X = \text{Spec}(k)$ and $G = \text{Gal}(k^{\text{sep}}|k)$ its absolute Galois group.

Theorem

- $Y \to X$ is étale $\iff Y = \text{Spec}(\prod_{i=1}^{r} L_i)$, where $L_i|k$ is a finite separable extension.
- The functor
  $$\tilde{X}_{\text{ét}} \to [\text{Continous } G\text{-sets}]$$
  $$F \mapsto \lim_{\longrightarrow} F(\text{Spec}(k'))$$

  $k \subset k' \subset k^{\text{sep}}, \text{ finite}$

  is an equivalence of categories.

- $$H^q(X_{\text{ét}}, F) \cong H^q(G, \lim_{\longrightarrow} F(\text{Spec}(k')))$$

- Here the right-hand side is the Galois-cohomology.
Étale cohomology yields the right cohomology theory for *torsion coefficients*.
More effort is needed for coefficients in a field with characteristic 0.

\( l \)-adic cohomology (\( l \neq \text{char} k \) prime):

\[
H^i(X, \mathbb{Z}_l) = \lim_{\leftarrow \nu} H^i(X_{\text{ét}}, \mathbb{Z}/l^\nu \mathbb{Z}),
\]

\[
H^i(X, \mathbb{Q}_l) = H^i(X, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.
\]
Properties of $l$-adic cohomology

**Theorem**

1. *The groups $H^i(X, \mathbb{Q}_l)$ are vector spaces over $\mathbb{Q}_l$.*
2. *If $X$ is proper over $k$, then they are finite dimensional.*
3. *Functoriality in $X$: if $f : X \to Y$ is a morphism, then it induces a homomorphism on the cohomologies:*
   \[
   f^* : H^i(Y, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l).
   \]
4. *$H^i(X, \mathbb{Q}_l) = 0$ for $i > 2 \dim X$.*
5. *K"unneth-formula is valid.*
Properties of $l$-adic cohomology

Theorem

6. There is a cup-product structure

$$H^i(X, \mathbb{Q}_l) \times H^j(X, \mathbb{Q}_l) \to H^{i+j}(X, \mathbb{Q}_l)$$

defined for all $i, j$.

7. Poincaré duality: if $X$ is smooth and proper over $k$, of dimension $n$, then $H^{2n}(X, \mathbb{Q}_l)$ is 1-dimensional, and the cup-product pairing is a perfect pairing for each $i, 0 \leq i \leq 2n$. 
Let $X$ be smooth and proper over $k$. Suppose $f : X \to X$ has only isolated fixed points, whose number is $L(f, X)$. Assume moreover, that for each fixed point $x \in X$, assume that the action of $1 - df$ on $\Omega^1_X$ is injective. Then

$$L(f, X) = \sum_{i=0}^{2n} (-1)^i \text{Tr}(f^* H^i(X, \mathbb{Q}_l)) .$$
Thank you for your attention!

Questions?