

## Solutions to November 2011 Problems

**Problem 1.** As a classroom assignment to the Grade 4 class, the teacher asked them to find the sum of the decimal digits of all the numbers from 1 to 2011, inclusive. Little Carlotta found the answer in a couple of minutes. What is that answer?

**Solution.** We solve a simpler problem, finding the sum of the decimal digits of all the numbers from 0 to 999. It is a little nicer to use 006 instead of plain 6, and 027 instead of 27. That makes no difference to the digit sum. Each of 0, 1, 2, ..., 9 occurs as the units digit 100 times, as the tens digit a hundred times, and as the hundreds digit 100 times. The sum  $0 + 1 + \dots + 9$  is 45. So the sum of all the units digits is 4500, as is the sum of the tens digits, as is the sum of the hundreds digits, for a total of 13500.

Roughly the same analysis works for 1000 to 1999, except for the 1000 additional 1s. The total is 14500.

Now comes the hard part, the numbers 2000 to 2011. By hand, we get a total of 72. Add up. We get 28072.

**Problem 2.** Let  $N(r)$  be the number of points  $(a, b)$  with integer coordinates inside or on the circle with center the origin and radius  $r$ . Show that there is a positive real number  $R$  such that  $3.14R^2 < N(R) < 3.15R^2$ .

**Solution.** Note that 3.14 underestimates  $\pi$ , while 3.15 overestimates it. The idea is that if  $R$  is large, then the number of points with integer coordinates which are on or inside the circle is in some sense very close to the area  $\pi R^2$  of the circle.

Divide the plane into  $1 \times 1$  squares by (i) drawing the  $x$ -axis, and all lines parallel to the  $x$ -axis and at integer distance from the  $x$ -axis. and (ii) doing the same with the  $y$ -axis. If  $(a, b)$  is a point with integer coordinates such that  $(a, b)$  lies on or inside the circle with radius  $r$ , take the square which has  $(a, b)$  as (say) its lower left-hand corner at  $(a, b)$ , and colour it red. The area of the red region is then exactly the number of points  $(a, b)$  with integer coordinates that are on or inside the circle. The intuition is that if  $r$  is large, then the area of the red region is not too far from the area of the circle.

We can now be fairly sloppy with our estimates. Note that the red region is entirely inside the circle with radius  $r + \sqrt{2}$ , and that if  $r > \sqrt{2}$ , the circle with radius  $r - \sqrt{2}$  is entirely red. We therefore conclude that

$$\pi(r - \sqrt{2})^2 \leq N(r) \leq \pi(r + \sqrt{2})^2.$$

So if we choose  $R$  such that (i)  $\pi(R - \sqrt{2})^2 > 3.14R^2$  and (ii)  $\pi(R + \sqrt{2})^2 < 3.15R^2$ , then the inequalities we want will hold. Intuitively, Inequality (i) is harder to manage, since 3.14 is closer to  $\pi$  than 3.15, so let's try to satisfy that one.

Inequality (i) can be rewritten as

$$\frac{(R - \sqrt{2})^2}{R^2} > \frac{3.14}{\pi},$$

or equivalently, after taking square roots, as

$$1 - \sqrt{2}R > \sqrt{\frac{3.14}{\pi}}.$$

the calculator gives that  $\sqrt{3.14/\pi}$  is about 0.9997465, so our inequality will hold if  $R > 5578.5134$ . For safety, and to guard against roundoff error, let's make  $R$  any number  $> 5579$ . It is not hard to verify that for such  $R$ , Inequality (ii) also holds. We omit the details, which are much like the ones that we used for Inequality (i). Actually, one can verify that Inequality (ii) already holds if  $R > 1058$ .

**Problem 3.** Find exact expressions for all the real roots of the equation

$$2\sqrt[3]{2x-1} = x^3 + 1.$$

**Solution.** This one is tricky. The root  $x = 1$  can be spotted by inspection, but what about other real roots? Graphing software will show that there are some.

It is all too easy to turn the relatively pleasant-looking equation into a mess. In particular, cubing both sides and expanding may be counterproductive. Let  $2y = x^3 + 1$ . Then  $\sqrt[3]{2x-1} = y$ , which (by cubing) is equivalent to  $2x - 1 = y^3$ , that is, to  $2x = y^3 + 1$ . So making  $x = y$  will give us solutions. We arrive at the equation  $2x = x^3 + 1$ , or equivalently  $x^3 - 2x + 1 = 0$ .

The equation has the obvious solution  $x = 1$ . So the polynomial  $x - 1$  is a factor of  $x^3 - 2x + 1$ . Do the division. We get

$$x^3 - 2x + 1 = (x - 1)(x^2 + x - 1).$$

Thus the roots of  $x^2 + x - 1 = 0$  are also solutions. These are  $x = \frac{-1 \pm \sqrt{5}}{2}$ .

Are there any other real solutions of the original equation? Look at the equations  $2y = x^3 + 1$  and  $2x = y^3 + 1$ . Subtract. We get  $2(y - x) = x^3 - y^3$ . Suppose that  $x \neq y$ . Note that  $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ . If  $x \neq y$ , we can divide both sides by  $x - y$ . We obtain  $-2 = x^2 + xy + y^2$ . This has no real solutions, for  $x^2 + xy + y^2 = (x + y/2)^2 + 3y^2/4$ , and  $(x + y/2)^2 + 3y^2/4$  can never be negative.

**Problem 4.** Let  $f(x) = (10 - x)(10 - \sqrt{92 - x^2})$ . Find the largest value and the smallest value taken on by  $f(x)$  as  $x$  ranges over the interval  $0 \leq x \leq 9$ . Ideally, the argument should not involve the calculus.

**Solution.** We give an elementary (meaning no calculus, but not at all easy) argument for both maximum and minimum.

Because it is pleasant to have symmetry for a while, let  $y = \sqrt{81 - x^2}$ . We want to study the behaviour of  $(10 - x)(10 - y)$ , that is, of

$$xy - 10(x + y) + 100. \quad (\text{Expression 1})$$

We will find the maximum and minimum values of Expression 1, given that  $x^2 + y^2 = 81$  and  $x \geq 0, y \geq 0$ .

Now, with some sadness, we break symmetry. So soon! Let  $w = x + y$  and  $xy = v$ . We are interested in the behaviour of

$$v - 10w + 100. \quad (\text{Expression 2})$$

But since  $(x + y)^2 - 2xy = x^2 + y^2$ , we have  $w^2 - 2v = 81$ . So we are interested in the behaviour of

$$\frac{w^2 - 81}{2} - 10w + 100, \quad \text{that is, of}$$
$$\frac{1}{2}(w^2 - 20w + 119).$$

Complete the square. We get

$$\frac{1}{2}((w - 10)^2 + 19). \quad (\text{Expression 3})$$

The analysis is now straightforward. There is a local minimum at  $w = 10$ . The minimum \*value\* is  $19/2$ .

The maximum is reached where  $w$ , that is,  $x + y$ , reaches a maximum subject to  $x^2 + y^2 = 81$ . Since  $(x + y)^2 + (x - y)^2 = 2(x^2 + y^2) = 162$ , the maximum value of  $(x + y)^2$ , and hence of  $x + y$ , occurs where  $x = y$ .

We can if we wish find the values of  $x$  at which we have a minimum. We need to solve the system  $x + y = 10$ ,  $x^2 + y^2 = 81$ . That gives  $2xy = 10^2 - 81 = 19$ , so  $(x - y)^2 = 81 - 19 = 62$ , and therefore  $x - y = \pm\sqrt{62}$ , and now we can find  $x$  and  $y$ .