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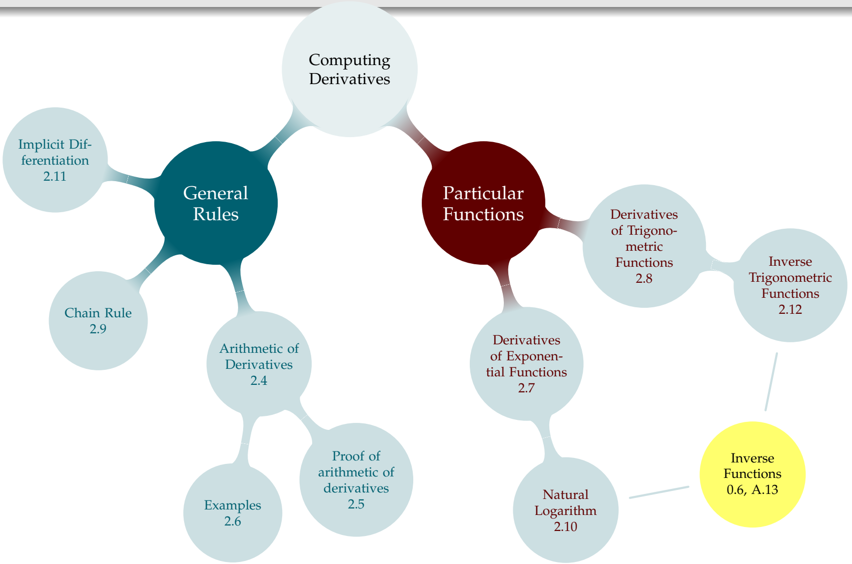


Invertibility Game

- A function $y = f(x)$ is known to both players
- Player A chooses a secret value x in the domain of $f(x)$
- Player A tells Player B what $f(x)$ is
- Player B tries to guess Player A's x-value.

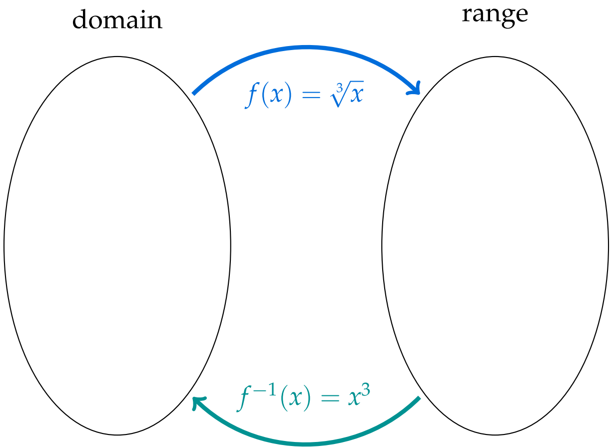
Round 1: $f(x) = 2x$

Round 2: $f(x) = \sqrt[3]{x}$

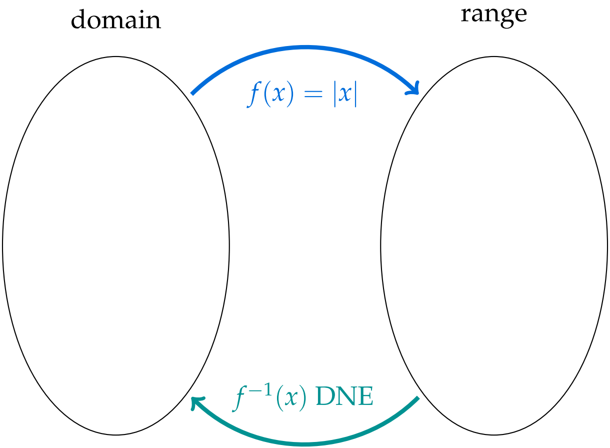
Round 3: $f(x) = |x|$

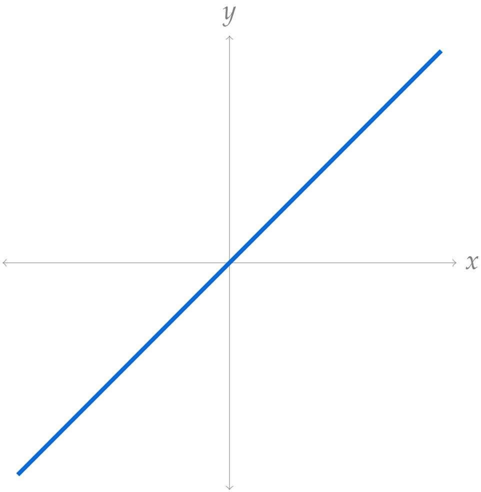
Round 4: $f(x) = \sin x$

FUNCTIONS ARE MAPS



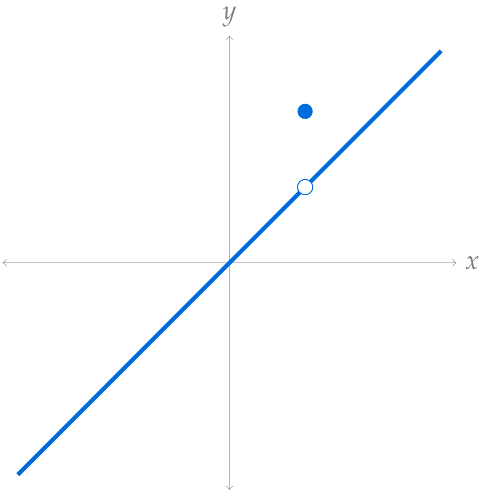
Functions are Maps



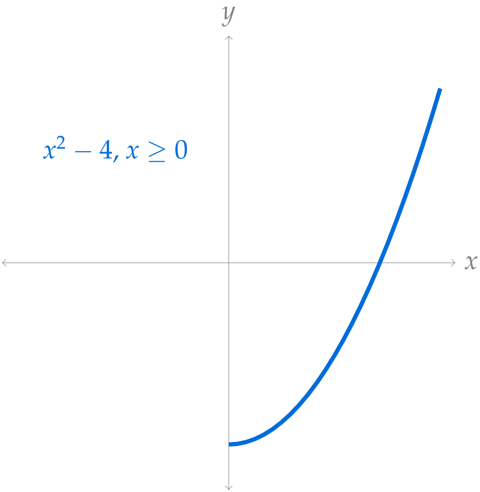


A. invertible

B. not invertible



A. invertible B. not invertible



A. invertible

B. not invertible

Relationship between $f(x)$ and $f^{-1}(x)$

Let f be an invertible function.
What is $f^{-1}(f(x))$?

A. x

B. 1

C. 0

D. not sure

Invertibility

In order for a function to be invertible , different x values cannot map to the same y value.
We call such a function **one-to-one**, or **injective**.

Suppose $f(x) = \sqrt[3]{19 + x^3}$. What is $f^{-1}(3)$? (simplify your answer)

What is $f^{-1}(10)$? (do not simplify)

What is $f^{-1}(x)$?

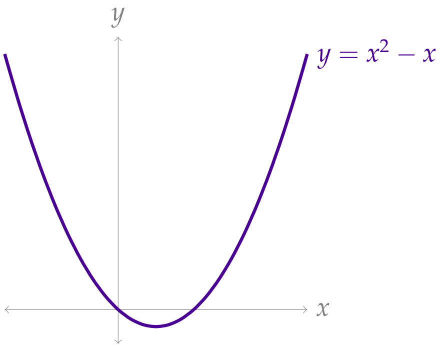
Let $f(x) = x^2 - x$.

1. Sketch a graph of $f(x)$, and choose a (large) domain over which it is invertible.

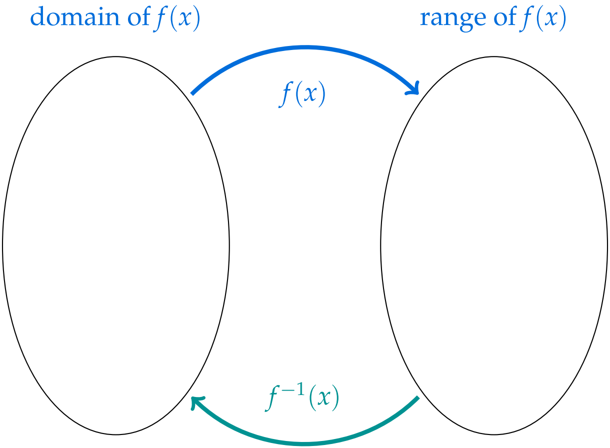
2. For the domain you chose, evaluate $f^{-1}(20)$.

3. For the domain you chose, evaluate $f^{-1}(x)$.

4. What are the domain and range of $f^{-1}(x)$? What are the (restricted) domain and range of $f(x)$?



y
$y = x^2 - x$
x


domain of $f(x)$
range of $f(x)$
$f(x)$
$f^{-1}(x)$

INVERTIBILITY GAME: $f(x) = e^x$ $\quad f^{-1}(x) = \log_e x$

- I'm thinking of an x. Your clue: $f(x) = e$. What is my x?
- I'm thinking of an x. Your clue: $f(x) = 1$. What is my x?
- I'm thinking of an x. Your clue: $f(x) = \frac{1}{e}$. What is my x?
- I'm thinking of an x. Your clue: $f(x) = e^3$. What is my x?
- I'm thinking of an x. Your clue: $f(x) = 0$. What is my x?

1. Suppose $0 < x < 1$. Then $\log_e(x)$ is...

2. Suppose $-1 < x < 0$. Then $\log_e(x)$ is...

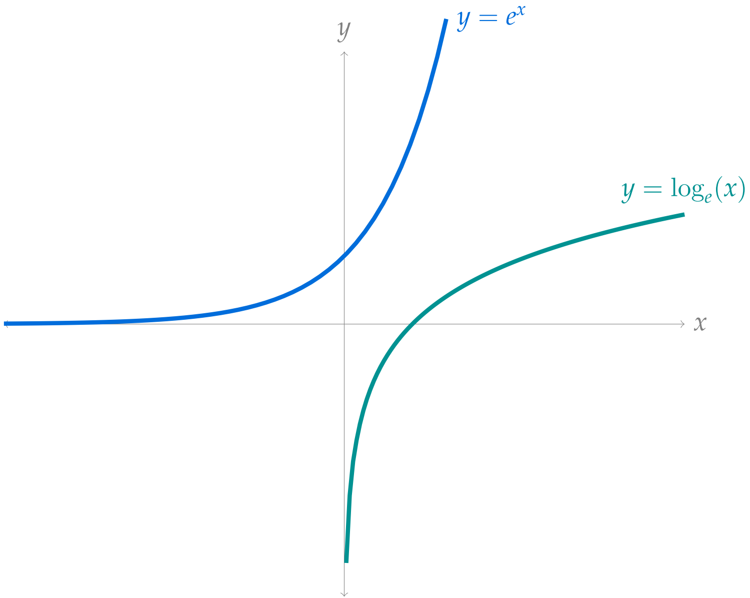
3. Suppose $e < x$. Then $\log_e(x)$ is...

A. positive
B. negative
C. greater than one
D. less than one
E. undefined

Exponents and Logarithms

$$f(x) = e^x \qquad\qquad f^{-1}(x) = \log_e(x) = \ln(x) = \log(x)$$

x	e^x	e fact $\leftrightarrow \log_e$ fact	x	$\log_e(x)$
0	1			
1	e			
-1	$\frac{1}{e}$			
n	e^n			



$y = e^x$
y
$y = \log_e(x)$
x

Logs of Other Bases: $\log_n(x)$ is the inverse of n^x

$\log_{10} 10^8 =$

A. 0

B. 8

C. 10

D. other

$\log_2 16 =$

A. 1

B. 2

C. 3

D. other

Logarithm Rules

Let A and B be positive, and let n be any real number.

$\log(A \cdot B) = \log(A) + \log(B)$

Proof: $\log(A \cdot B) = \log(e^{\log A} e^{\log B}) = \log(e^{\log A + \log B}) = \log(A) + \log(B)$

$\log(A/B) = \log(A) - \log(B)$

Proof: $\log(A/B) = \log\left(\frac{e^{\log A}}{e^{\log B}}\right) = \log(e^{\log A - \log B}) = \log A - \log B$

$\log(A^n) = n \log(A)$

Proof: $\log(A^n) = \log\left(\left(e^{\log A}\right)^n\right) = \log\left(e^{n \log A}\right) = n \log A$

Logarithm Rules

Let A and B be positive, and let n be any real number.
$\log(A \cdot B) = \log(A) + \log(B)$
$\log(A/B) = \log(A) - \log(B)$
$\log(A^n) = n\log(A)$

Write as a single logarithm:

$$f(x) = \log\left(\frac{10}{x^2}\right) + 2\log x + \log(10+x)$$

Base Change

$$\begin{aligned} \text{Fact:} \qquad b^{\log_b(a)} &= a \\ \Rightarrow \log\left(b^{\log_b(a)}\right) &= \log(a) \\ \Rightarrow \log_b(a)\log(b) &= \log(a) \\ \Rightarrow \log_b(a) &= \frac{\log(a)}{\log(b)} \end{aligned}$$

In general, for positive a, b, and c:

$$\boxed{\log_b(a) = \frac{\log_c(a)}{\log_c(b)}}$$

In general, for positive a, b, and c:

$$\boxed{\log_b(a) = \frac{\log_c(a)}{\log_c(b)}}$$

Suppose your calculator can only compute logarithms base 10. What would you enter to calculate $\log(17)$?

Suppose your calculator can only compute natural logarithms. What would you enter to calculate $\log_2(57)$?

Suppose your calculator can only compute logarithms base 2. What would you enter to calculate $\log(2)$?

Decibels: For a particular measure of the power P of a sound wave, the decibels of that sound is:

$$10\log_{10}(P)$$

So, every ten decibels corresponds to a sound being ten **times** louder.

A lawnmower emits a 100dB sound. How much sound will two lawnmowers make?

A. 100 dB
B. 110 dB
C. 200 dB
D. other

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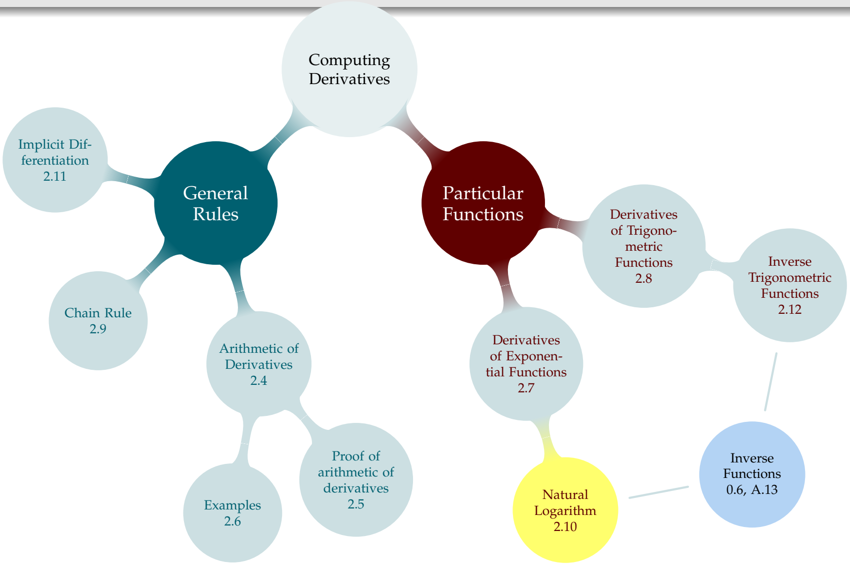

Differentiating the Natural Logarithm

Calculate $\frac{\mathrm{d}}{\mathrm{d}x}\{\log_e x\}$.
One Weird Trick:

$$
\begin{aligned}
x &= e^{\log_e x} \\
\frac{\mathrm{d}}{\mathrm{d}x}\{x\} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{e^{\log_e x}\right\} \\
1 &= e^{\log_e x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\{\log_e x\} = x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\{\log_e x\} \\
\frac{1}{x} &= \frac{\mathrm{d}}{\mathrm{d}x}\{\log_e x\}
\end{aligned}
$$

Derivative of Natural Logarithm

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\log_e |x|\} = \frac{1}{x} \qquad (x \neq 0)$$

Differentiate: $f(x) = \log_e |x^2 + 1|$

Derivatives of Logarithms – Corollary 2.10.6

For $a > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log_a |x|] = \frac{1}{x \log a}$$

In particular:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log |x|] = \frac{1}{x}$$

Differentiate: $f(x) = \log_e |\cot x|$

LOGARITHMIC DIFFERENTIATION - A FANCY TRICK

- $\log(f \cdot g) = \log f + \log g$

  multiplication turns into addition

- $\log\left(\frac{f}{g}\right) = \log f - \log g$

  division turns into subtraction

- $\log\left(f^g\right) = g \log f$

  exponentiation turns into multiplication

  We can exploit these properties to differentiate!

Logarithmic Differentiation

In general, if $f(x) \neq 0$, $\frac{\mathrm{d}}{\mathrm{d}x}\left[\log|f(x)|\right] = \frac{f'(x)}{f(x)}$.

$$f(x) = \left(\frac{(2x+5)^4(x^2+1)}{x+3}\right)^5$$

Find $f'(x)$.

Logarithmic Differentiation - A Fancy Trick

$$f(x) = \qquad \left(\frac{(2x+5)^4(x^2+1)}{x+3}\right)^5$$

Logarithmic Differentiation - A Fancy Trick

Differentiate:

$$f(x) = x^x$$

Logarithmic Differentiation - A Fancy Trick

Differentiate:

$$f(x) = \left(\frac{(x^{15} - 9x^2)^{10}(x + x^2 + 1)}{(x^7 + 7)(x + 1)(x + 2)(x + 3)} \right)^5$$

$$f(x) = \frac{(x^8 - e^x)(\sqrt{x} + 5)}{\csc^5 x}$$

$$f(x) = (x^2 + 17)(32x^5 - 8)(x^{98} - x^{57} + 32x^2)^4(32x^{10} - 10x^{32})$$

Find $f'(x)$.

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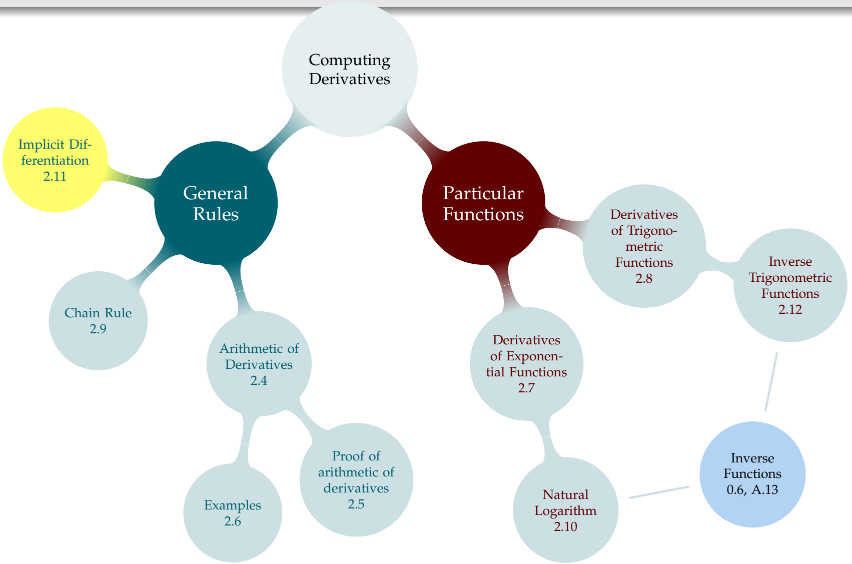


Implicitly Defined Functions

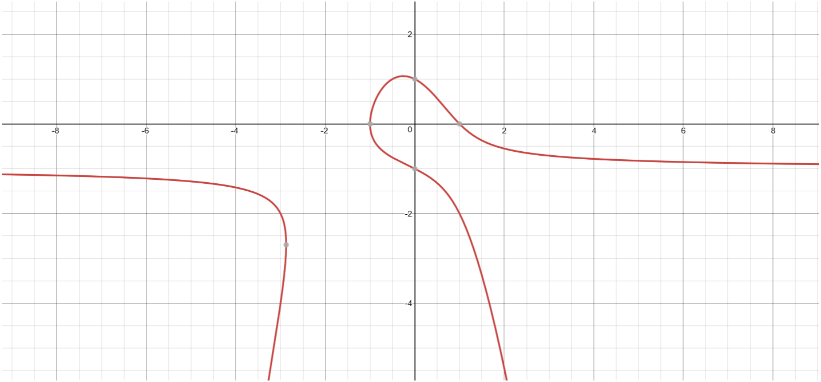
$$y^2 + x^2 + xy + x^2y = 1$$

Which of the following points are on the curve?

$(0, 1)$, $(0, -1)$, $(0, 0)$, $(1, 1)$

If $x = -3$, what is y?

$$y^2 + x^2 + xy + x^2y = 1$$



Still has a slope: $\frac{\Delta y}{\Delta x}$
Locally, y is still a function of x.

$$y^2 + x^2 + xy + x^2y = 1$$

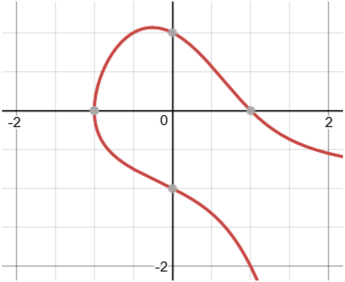
Consider y as a function of x. Can we find $\frac{dy}{dx}$?

$\frac{d}{dx}[y] =$ $\frac{d}{dx}[x] =$ $\frac{d}{dx}[1] =$

$$y^2 + x^2 + xy + x^2y = 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x + y + 2xy}{2y + x + x^2}$$

Necessarily, $\frac{\mathrm{d}y}{\mathrm{d}x}$ depends on **both** y and x. Why?



Now You

Suppose $x^4y + y^4x = 2$. Find $\frac{dy}{dx}$ at the point $(1, 1)$.

Now You Suppose $\dfrac{3y^2 + 2y + y^3}{x^2 + 1} = x$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 0$, and the equations of the associated tangent line(s).

Use implicit differentiation to differentiate $\log(x)$, $x > 0$.

$$\log x = y(x)$$
$$x = e^{y(x)}$$

Use implicit differentiation to differentiate $\log|x|$, $x < 0$.

Use implicit differentiation to differentiate $\log_a(x)$, where $a > 0$ is a constant and $x > 0$.

Use implicit differentiation to differentiate $\log_a |x|$, $a > 0$.

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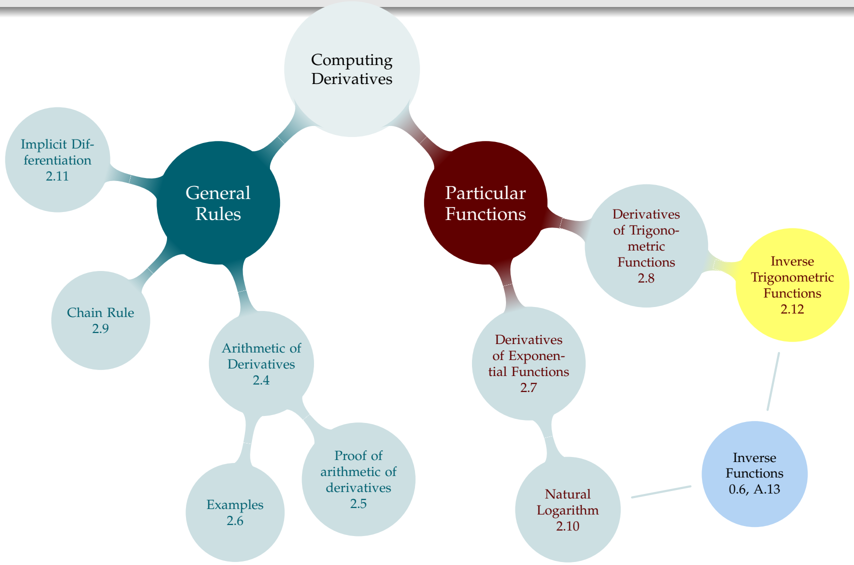
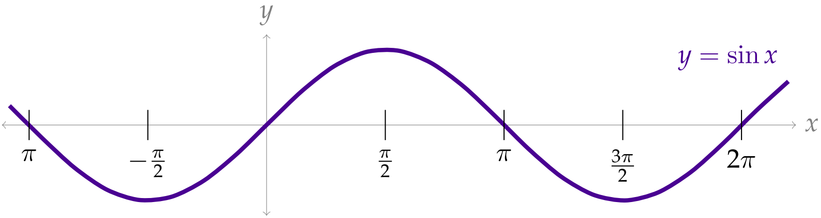


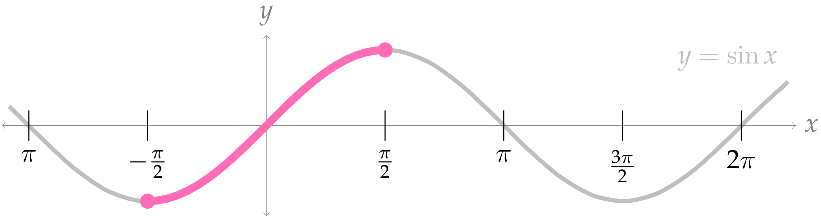
Invertibility Game



I'm thinking of a number x. Your hint: $\sin(x) = 0$. What number am I thinking of?

I'm thinking of a number x, and x is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Your hint: $\sin(x) = 0$. What number am I thinking of?

Arcsine



$\arcsin(x)$ is the inverse of $\sin x$ restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\arcsin x$ is the (unique) number θ such that:

- $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, and
- $\sin \theta = x$

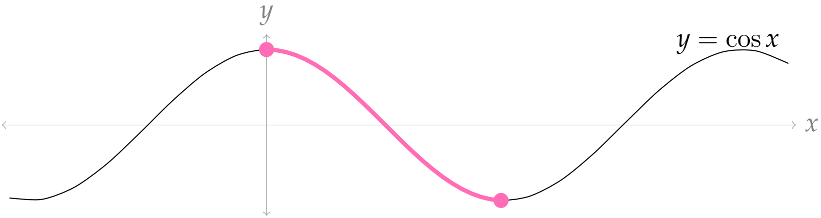
ARCSINE

Reference Angles:

θ	$\sin\theta$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1

- $\arcsin(0)$
- $\arcsin\left(\frac{1}{\sqrt{2}}\right)$
- $\arcsin\left(-\frac{1}{\sqrt{2}}\right)$
- $\arcsin\left(\frac{\pi}{2}\right)$
- $\arcsin\left(\frac{\pi}{4}\right)$

ARCCOSINE

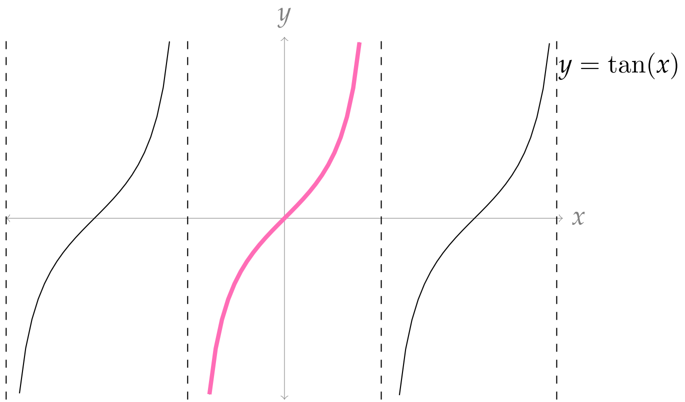


$\arccos(x)$ is the inverse of $\cos x$ restricted to $[0, \pi]$.

$\arccos(x)$ is the (unique) number θ such that:

- $\cos(\theta) = x$ and
- $0 \leq \theta \leq \pi$

Arctangent



$\arctan(x) = \theta$ means:
(1) $\tan(\theta) = x$ and
(2) $-\pi/2 < \theta < \pi/2$

Arcsecant, arcsine, and arccotangent

$$\operatorname{arcsec}(x) =$$

Arcsecant, arcsine, and arccotangent

$$\operatorname{arccsc}(x) = \arcsin\left(\tfrac{1}{x}\right)$$

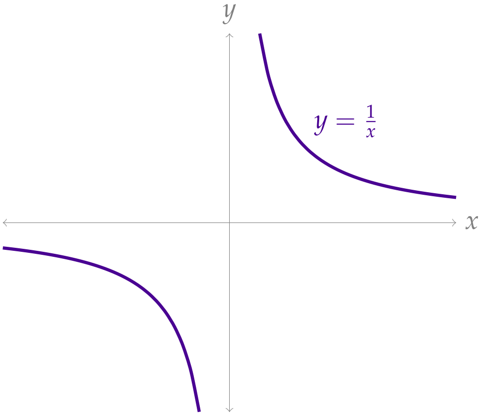
$$\begin{aligned}
\operatorname{arccsc}(x) &= y \\
\csc y &= x \\
\frac{1}{\sin y} &= x \\
\sin y &= \frac{1}{x} \\
y &= \arcsin\left(\tfrac{1}{x}\right) \\
\operatorname{arccsc}(x) &= \arcsin\left(\tfrac{1}{x}\right)
\end{aligned}$$

$$\operatorname{arccot}(x) = \arctan\left(\tfrac{1}{x}\right)$$

$$\begin{aligned}
\operatorname{arccot}(x) &= y \\
\cot y &= x \\
\frac{1}{\tan y} &= x \\
\tan y &= \tfrac{1}{x} \\
y &= \arctan\left(\tfrac{1}{x}\right) \\
\operatorname{arccot}(x) &= \arctan\left(\tfrac{1}{x}\right)
\end{aligned}$$

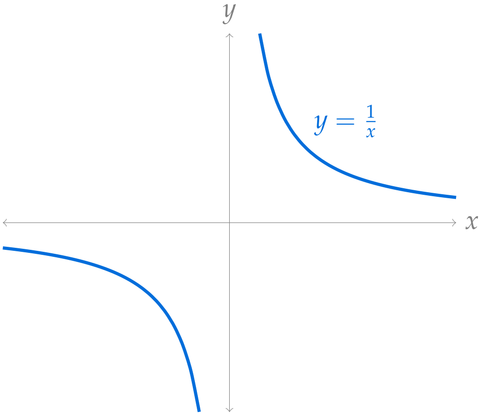
$$\text{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$$

The domain of $\arccos(y)$ is $-1 \leq y \leq 1$, so the domain of $\text{arcsec}(y)$ is



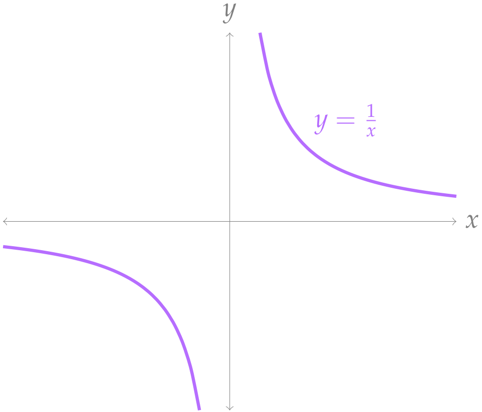
$$\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$$

Domain of $\arcsin(y)$ is $-1 \le y \le 1$, so the domain of $\operatorname{arccsc}(x)$ is



$$\text{arccot}(x) = \arctan\left(\frac{1}{x}\right)$$

Domain of $\arctan(x)$ is all real numbers, so the domain of $\text{arccot}(x)$ is



$$y = \arcsin x$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$y = \arctan x$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$y = \arccos x$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

To differentiate arcsecant, arccosecant, and arccotangent, you can use the chain rule!

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\operatorname{arccsc}(x)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\arcsin\left(\frac{1}{x}\right)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\arcsin\left(x^{-1}\right)\right]$$

Derivatives of Inverse Trigonometric Functions – Theorem 2.12.7

Memorize:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\arcsin x] = \frac{1}{1+x^2}$$

Be able to derive:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arccsc} x] = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arcsec} x] = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{arccot} x] = -\frac{1}{1+x^2}$$

Included Work

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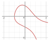
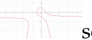
 screenshot of graph using Desmos Graphing Calculator, https://www.desmos.com/calculator (accessed 19 October 2017), 38

 screenshot of graph using Desmos Graphing Calculator, https://www.desmos.com/calculator (accessed 19 October 2017), 36