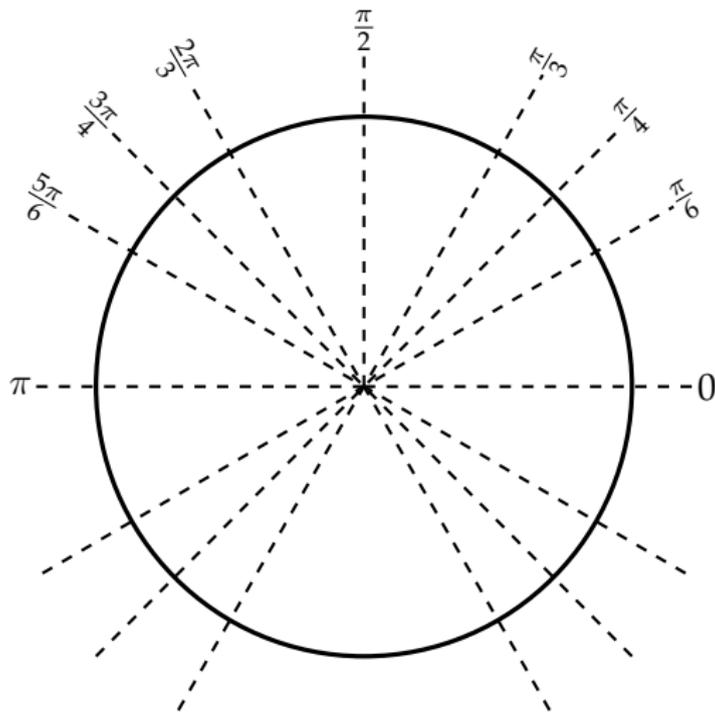


COMMONLY USED FACTS

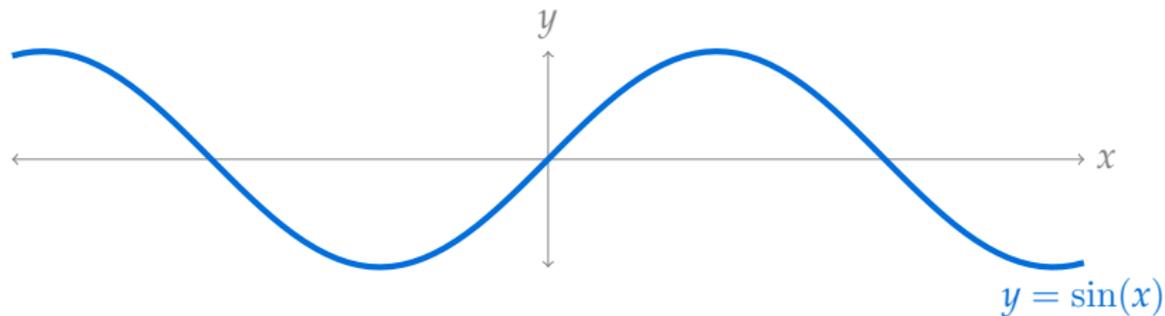
- ▶ Graphs of sine, cosine, tangent
- ▶ Sine, cosine, and tangent of reference angles: $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$
- ▶ How to use reference angles to find sine, cosine and tangent of other angles
- ▶ Identities: $\sin^2 x + \cos^2 x = 1;$ $\tan^2 x + 1 = \sec^2 x;$
 $\sin^2 x = \frac{1 - \cos(2x)}{2};$ $\cos^2 x = \frac{1 + \cos 2x}{2}$
- ▶ Conversion between radians and degrees

CLP-1 has an appendix on high school trigonometry that you should be familiar with.

REFERENCE ANGLES



DERIVATIVE OF SINE

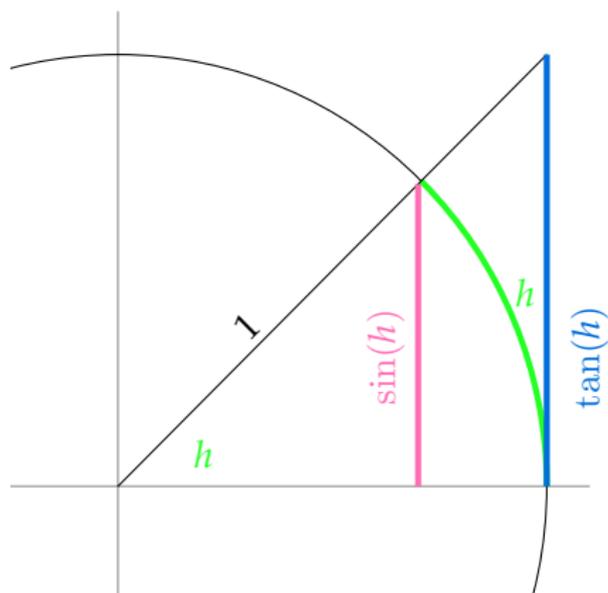


Consider the derivative of $f(x) = \sin(x)$.



$$\begin{aligned}
 \frac{d}{dx} \{\sin x\} &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x)(\cos(h) - 1)}{h} + \lim_{h \rightarrow 0} \frac{\cos(x)\sin(h)}{h} \\
 &= \sin(x) \lim_{h \rightarrow 0} \frac{\cos(0+h) - \cos(0)}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\
 &= \sin(x) \left. \frac{d}{dx} \{\cos(x)\} \right|_{x=0} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} = \boxed{\cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h}}
 \end{aligned}$$

since $\cos(x)$ has a horizontal tangent, and hence has derivative zero, at $x = 0$.



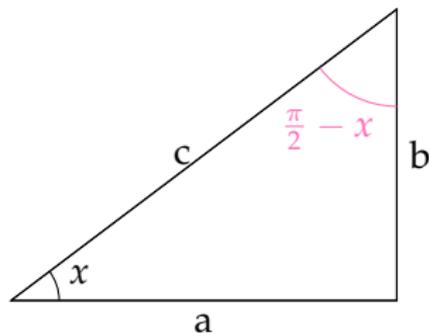
DERIVATIVES OF SINE AND COSINE

From before,

$$\frac{d}{dx}\{\sin(x)\} = \cos(x) \cdot \lim_{h \rightarrow 0} \frac{\sin(h)}{h} = \cos(x)$$

DERIVATIVE OF COSINE

Now for the derivative of \cos . We already know the derivative of \sin , and it is easy to convert between \sin and \cos using trig identities.



$$\sin x = \frac{b}{c} = \cos\left(\frac{\pi}{2} - x\right)$$
$$\cos x = \frac{a}{c} = \sin\left(\frac{\pi}{2} - x\right)$$

When we use radians:

Derivatives of Trig Functions

$$\frac{d}{dx} \{\sin(x)\} = \cos(x)$$

$$\frac{d}{dx} \{\cos(x)\} = -\sin(x)$$

$$\frac{d}{dx} \{\tan(x)\} =$$

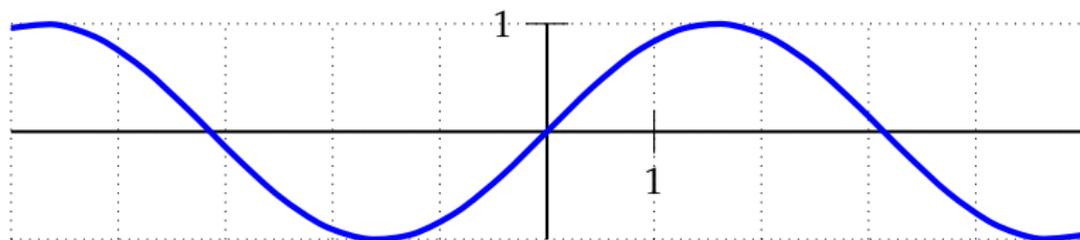
$$\frac{d}{dx} \{\sec(x)\} =$$

$$\frac{d}{dx} \{\csc(x)\} =$$

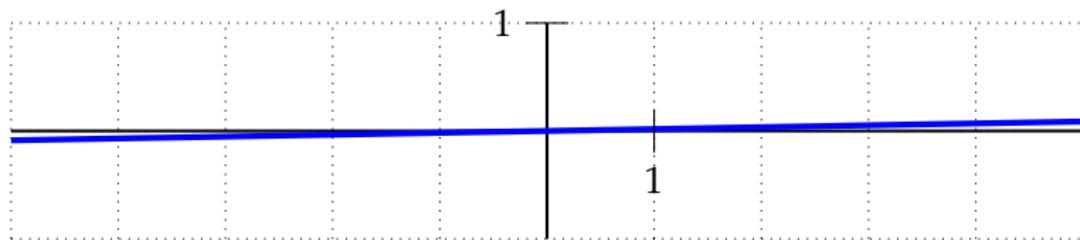
$$\frac{d}{dx} \{\cot(x)\} =$$

Honorable Mention

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$



$$y = \sin x, \text{ radians}$$



$$y = \sin x, \text{ degrees}$$

OTHER TRIG FUNCTIONS

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

OTHER TRIG FUNCTIONS

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\begin{aligned}\frac{d}{dx}[\sec(x)] &= \frac{d}{dx} \left[\frac{1}{\cos(x)} \right] \\ &= \frac{\cos(x)(0) - (1)(-\sin(x))}{\cos^2(x)} \\ &= \frac{\sin(x)}{\cos^2(x)} \\ &= \frac{1}{\cos(x)} \frac{\sin(x)}{\cos(x)} \\ &= \sec(x) \tan(x)\end{aligned}$$

OTHER TRIG FUNCTIONS

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\begin{aligned}\frac{d}{dx}[\csc(x)] &= \frac{d}{dx} \left[\frac{1}{\sin(x)} \right] \\ &= \frac{\sin(x)(0) - (1)\cos(x)}{\sin^2(x)} \\ &= \frac{-\cos(x)}{\sin^2(x)} \\ &= \frac{-1}{\sin(x)} \frac{\cos(x)}{\sin(x)} \\ &= -\csc(x) \cot(x)\end{aligned}$$

OTHER TRIG FUNCTIONS

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$\begin{aligned}\frac{d}{dx}[\cot(x)] &= \frac{d}{dx} \left[\frac{\cos(x)}{\sin(x)} \right] \\ &= \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} \\ &= \frac{-1}{\sin^2(x)} \\ &= -\csc^2(x)\end{aligned}$$

MEMORIZE

$$\frac{d}{dx}\{\sin(x)\} = \cos(x)$$

$$\frac{d}{dx}\{\cos(x)\} = -\sin(x)$$

$$\frac{d}{dx}\{\tan(x)\} = \sec^2(x)$$

$$\frac{d}{dx}\{\sec(x)\} = \sec(x)\tan(x)$$

$$\frac{d}{dx}\{\csc(x)\} = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\{\cot(x)\} = -\csc^2(x)$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Let $f(x) = \frac{x \tan(x^2 + 7)}{15e^x}$. Use the definition of the derivative to find $f'(0)$.

Differentiate $(e^x + \cot x)(5x^6 - \csc x)$.

$$\text{Let } h(x) = \begin{cases} \frac{\sin x}{x} & , \quad x < 0 \\ \frac{ax+b}{\cos x} & , \quad x \geq 0 \end{cases}$$

Which values of a and b make $h(x)$ continuous at $x = 0$?

Practice and Review

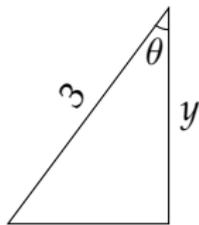
$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Is $f(x)$ differentiable at $x = 0$?

$$g(x) = \begin{cases} e^{\frac{\sin x}{x}} & , \quad x < 0 \\ (x - a)^2 & , \quad x \geq 0 \end{cases}$$

What value(s) of a makes $g(x)$ continuous at $x = 0$?

A ladder 3 meters long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall, measured in radians, and let y be the height of the top of the ladder. If the ladder slides away from the wall, how fast does y change with respect to θ ?
When is the top of the ladder sinking the fastest? The slowest?



Suppose a point in the plane that is r centimetres from the origin, at an angle of θ ($0 \leq \theta \leq \frac{\pi}{2}$), is rotated $\pi/2$ radians. What is its new coordinate (x, y) ? If the point rotates at a constant rate of a radians per second, when is the x coordinate changing fastest and slowest with respect to θ ?

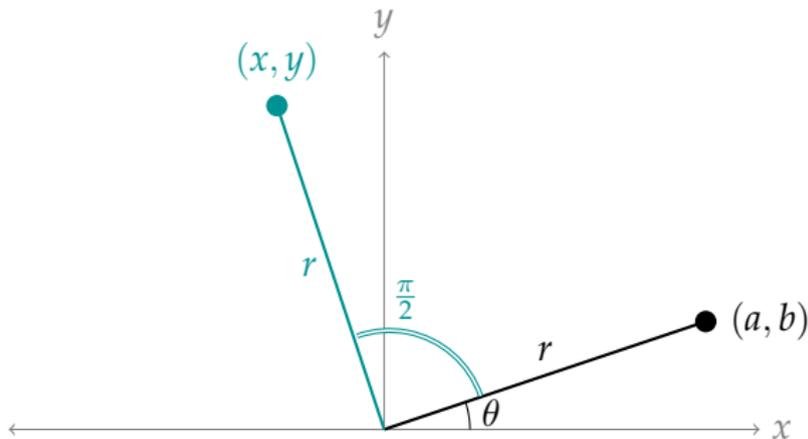
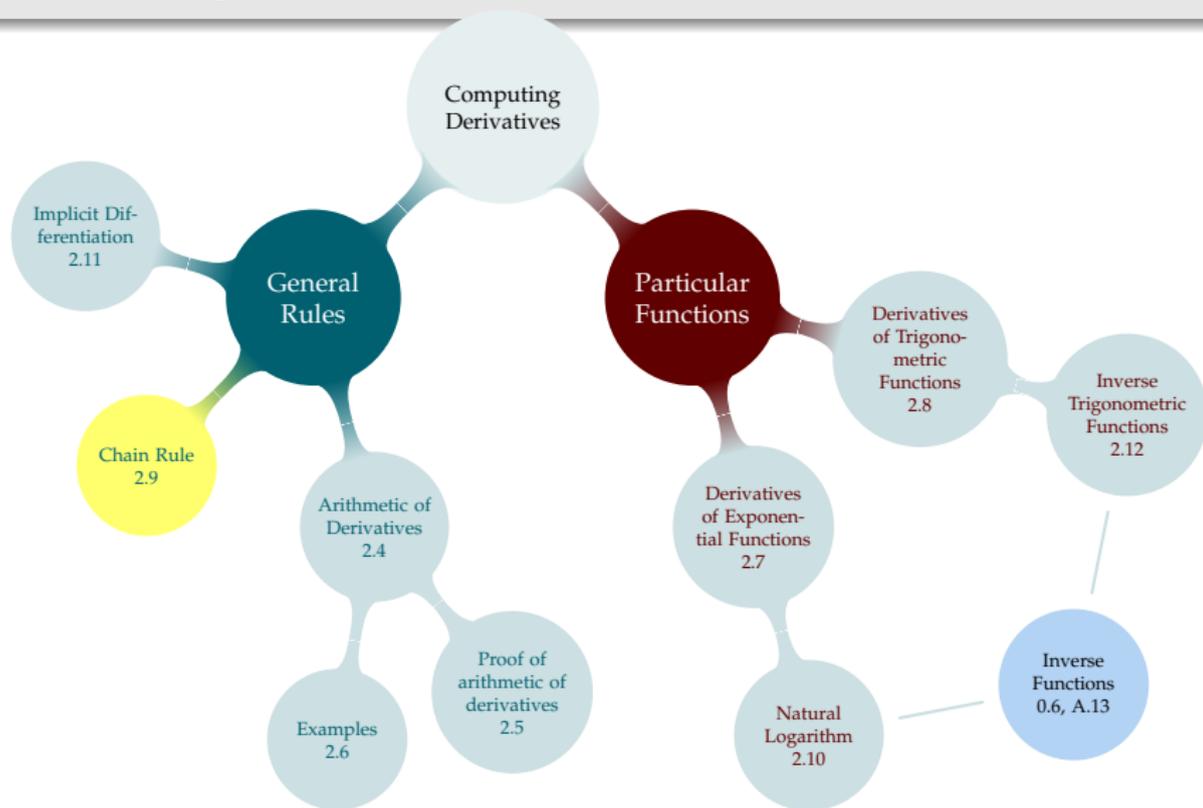
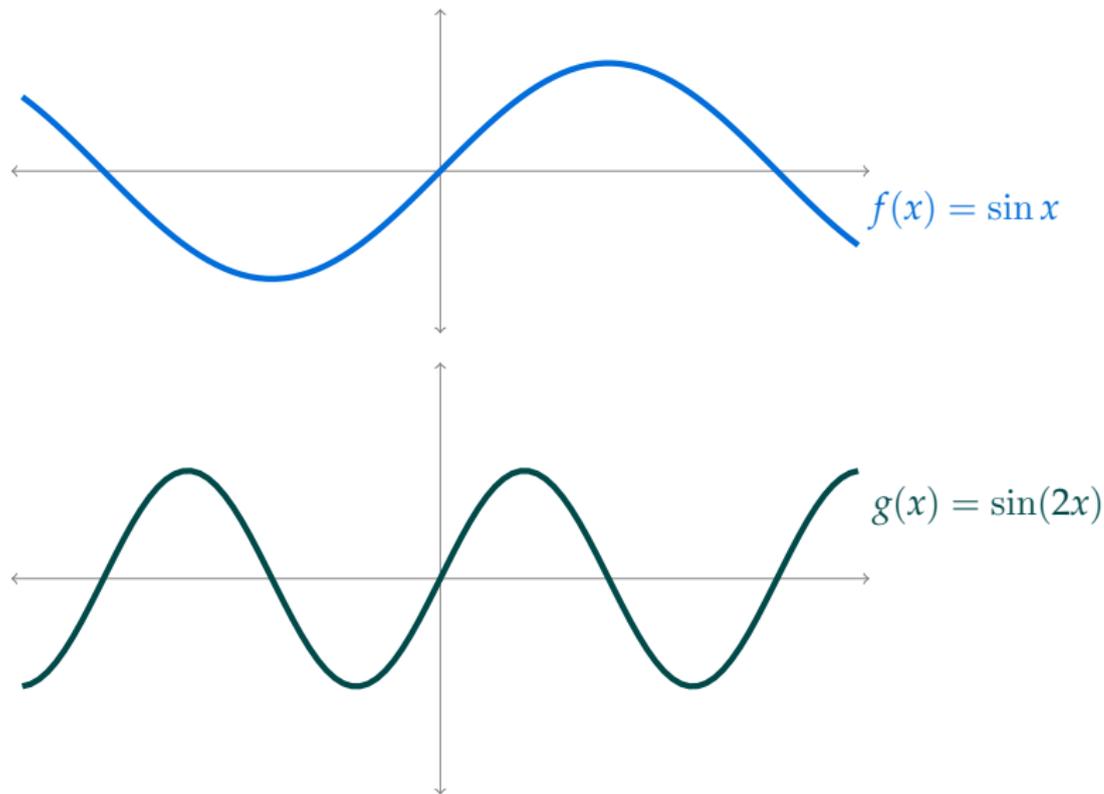


TABLE OF CONTENTS



INTUITION: $\sin x$ VERSUS $\sin(2x)$



COMPOUND FUNCTIONS

Video: 2:27-3:50

Morton, Jennifer. (2014). *Balancing Act: Otters, Urchins and Kelp*.
Available from [https://www.kqed.org/quest/67124/
balancing-act-otters-urchins-and-kelp](https://www.kqed.org/quest/67124/balancing-act-otters-urchins-and-kelp)

KELP POPULATION

- k kelp population
- u urchin population
- o otter population
- p public policy

$k(u)$

$k(u(o))$

$k(u(o(p)))$

These are examples of compound functions.

Should $\frac{d}{do}k(u(o))$ be positive or negative?

- A. positive
- B. negative
- C. I'm not sure

Should $k'(u)$ be positive or negative?

- A. positive
- B. negative
- C. I'm not sure

DIFFERENTIATING COMPOUND FUNCTIONS

$$\begin{aligned}\frac{d}{dx}\{f(g(x))\} &= \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{h} \left(\frac{g(x+h) - g(x)}{g(x+h) - g(x)} \right) \\ &= \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(\boxed{g(x+h)}) - f(\boxed{g(x)})}{\boxed{g(x+h)} - \boxed{g(x)}} \cdot g'(x)\end{aligned}$$

Set $H = g(x+h) - g(x)$. As $h \rightarrow 0$, we also have $H \rightarrow 0$. So

$$\begin{aligned}&= \lim_{H \rightarrow 0} \frac{f(g(x) + H) - f(g(x))}{H} \cdot g'(x) \\ &= f'(g(x)) \cdot g'(x)\end{aligned}$$

CHAIN RULE

Chain Rule – Theorem 2.9.3

Suppose f and g are differentiable functions. Then

$$\frac{d}{dx}\{f(g(x))\} = f'(g(x))g'(x) = \frac{df}{dg}(g(x))\frac{dg}{dx}(x)$$

In the case of kelp, $\frac{d}{do}k(u(o)) = \frac{dk}{du}(u(o))\frac{du}{do}(o)$

Chain Rule

Suppose f and g are differentiable functions. Then

$$\frac{d}{dx}\{f(g(x))\} = f'(g(x))g'(x) = \frac{df}{dg}(g(x))\frac{dg}{dx}(x)$$

Example: suppose $F(x) = \sin(e^x + x^2)$.

$$F(v) = \left(\frac{v}{v^3 + 1} \right)^6$$

NOW
YOU



Let $f(x) = (10^x + \csc x)^{1/2}$. Find $f'(x)$.

NOW
YOU



Suppose $o(t) = e^t$, $u(o) = \frac{1}{o + \sin(o)}$, and $t \geq 10$ (so all

these functions are defined). Using the chain rule, find $\frac{d}{dt} u(o(t))$.

Note: your answer should depend only on t : not o .

Evaluate $\frac{d}{dx} \left\{ x^2 + \sec \left(x^2 + \frac{1}{x} \right) \right\}$

Evaluate $\frac{d}{dx} \left\{ \frac{1}{x + \frac{1}{x + \frac{1}{x}}} \right\}$

Included Work



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