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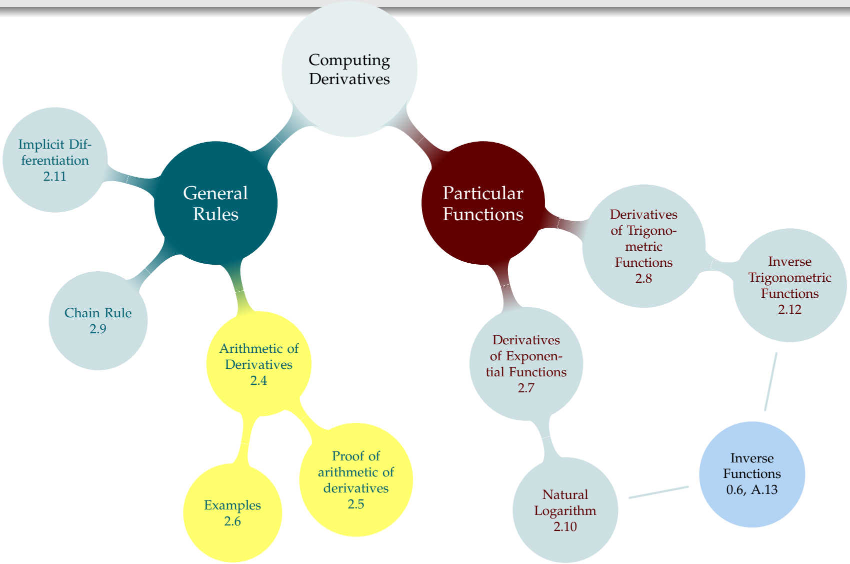


Derivatives of Lines

$$f(x) = 2x - 15$$

The equation of the tangent line to $f(x)$ at $x = 100$ is:

$f'(1) =$

A. 0 B. 1 C. 2 D. -15 E. -13

$f'(5) =$

$f'(-13) =$

$$g(x) = 13$$

$g'(1) =$

A. 0 B. 1 C. 2 D. 13

Adding a Constant

Adding or subtracting a constant to a function does not change its derivative.

We saw

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(3-0.8t^2\right)\Big|_{t=1} = -1.6$$

So,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(10-0.8t^2\right)\Big|_{t=1} =$$

Differentiating Sums

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{f(x)+g(x)\right\} =$$

Constant Multiple of a Function

Let a be a constant.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{a \cdot f(x)\right\} =$$

Rules – Lemma 2.4.1

Suppose $f(x)$ and $g(x)$ are differentiable, and let c be a constant number. Then:

- $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)+g(x)\} = f'(x)+g'(x)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)-g(x)\} = f'(x)-g'(x)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\{cf(x)\} = cf'(x)$

For instance: let $f(x) = 10\left((2x-15)+13-\sqrt{x}\right)$. Then $f'(x) =$

Now You

Suppose $f'(x) = 3x$, $g'(x) = -x^2$, and $h'(x) = 5$.

Calculate:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{f(x) + 5g(x) - h(x) + 22\right\}$$

A. $3x - 5x^2$

B. $3x - 5x^2 - 5$

C. $3x - 5x^2 - 5 + 22$

D. none of the above

Derivatives of Products

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x\} = 1$$

True or False:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\{2x\} &= \frac{\mathrm{d}}{\mathrm{d}x}\{x+x\}\\ &= [1]+[1]\\ &= 2\end{aligned}$$

True or False:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\{x^2\} &= \frac{\mathrm{d}}{\mathrm{d}x}\{x\cdot x\}\\ &= [1]\cdot[1]\\ &= 1\end{aligned}$$

What to do with Products?

Suppose $f(x)$ and $g(x)$ are differentiable functions of x. What about $f(x)g(x)$?

Product Rule – Theorem 2.4.3

For differentiable functions $f(x)$ and $g(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$$

Example:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\right] =$$

Example: suppose $f(x) = 3x^2, f'(x) = 6x$, $g(x) = \sin(x), g'(x) = \cos(x)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[3x^2\sin(x)\right] =$$

Given $\frac{d}{dx}[2x+5]=2,\qquad \frac{d}{dx}\left[\sin(x^2)\right]=2x\cos(x^2),\qquad \frac{d}{dx}\left[x^2\right]=2x$

Now You

$$f(x)=(2x+5)\sin(x^2)$$

A. $f'(x)=(2)\left(2x\cos(x^2)\right)(2x)$
B. $f'(x)=(2)\left(2x\cos(x^2)\right)$
C. $f'(x)=(2x+5)(2)+\sin(x^2)\left(2x\cos(x^2)\right)$
D. $f'(x)=(2x+5)\left(2x\cos(x^2)\right)+(2)\sin(x^2)$
E. none of the above

NOW YOU

$$f(x) = a(x) \cdot b(x) \cdot c(x)$$

What is $f'(x)$?

Quotient Rule – Theorem 2.4.5

Let $f(x)$ and $g(x)$ be differentiable and $g(x) \neq 0$. Then:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Mnemonic: Low d'high minus high d'low over lowlow.

Quotient Rule – Theorem 2.4.5

Let $f(x)$ and $g(x)$ be differentiable and $g(x) \neq 0$. Then:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Mnemonic: Low d'high minus high d'low over lowlow.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{2x+5}{3x-6}\right\} =$$

Quotient Rule – Theorem 2.4.5

Let $f(x)$ and $g(x)$ be differentiable and $g(x) \neq 0$. Then:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Mnemonic: Low d'high minus high d'low over lowlow.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{5x}{\sqrt{x}-1}\right\} =$$

Now You

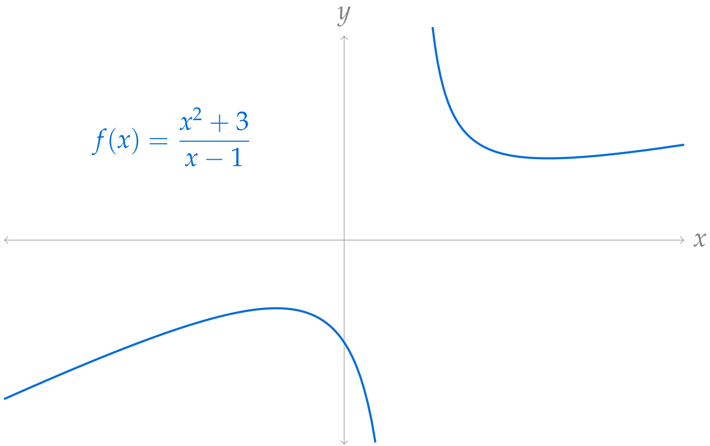
Differentiate the following.

$$f(x) = 2x + 5$$
$$g(x) = (2x + 5)(3x - 7) + 25$$
$$h(x) = \frac{2x + 5}{8x - 2}$$
$$j(x) = \left(\frac{2x + 5}{8x - 2}\right)^2$$

Rules

Product: $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)g(x)\} = f(x)g'(x) + g(x)f'(x)$

Quotient: $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

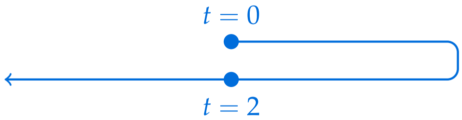


For which values of x is the tangent line to the curve horizontal?

The position of an object moving left and right at time t, $t \geq 0$, is given by

$$s(t) = -t^2(t-2)$$

where a positive position means it is to the right of its starting position, and a negative position means it is to the left. First it moves to the right, then it moves left forever.



What is the farthest point to the right that the object reaches?

More About the Product Rule

$$\frac{d}{dx}\{x^2\} = \frac{d}{dx}\{x \cdot x\} = x(1) + x(1)$$
$$= 2x$$

$$\frac{d}{dx}\{x^3\} = \frac{d}{dx}\{x \cdot x^2\}$$
$$= (x)(2x) + (x^2)(1) = 3x^2$$

$$\frac{d}{dx}\{x^4\} = \frac{d}{dx}\{x \cdot x^3\}$$
$$= x(3x^2) + x^3(1) = 4x^3$$

Where are these functions defined?

function	derivative
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^{30}	$30x^{29}$
x^n	nx^{n-1}

Cautionary Tale

With *functions* raised to a power, it's more complicated.

Differentiate $(2x+1)^2$

Power Rule – Corollary 2.6.17

$\frac{\mathrm{d}}{\mathrm{d}x}\{x^a\} = ax^{a-1}$ (where defined)

$$\frac{\mathrm{d}}{\mathrm{d}x}\{3x^5 + 7x^2 - x + 15\} =$$

Power Rule – Corollary 2.6.17

$\frac{\mathrm{d}}{\mathrm{d}x}\{x^a\} = ax^{a-1}$ (where defined)

Differentiate $\dfrac{(x^4+1)(\sqrt[3]{x}+\sqrt[4]{x})}{2x+5}$

Power Rule – Corollary 2.6.17

$\frac{\mathrm{d}}{\mathrm{d}x}\{x^a\} = ax^{a-1}$ (where defined)

Suppose a motorist is driving their car, and their position is given by $s(t) = 10t^3 - 90t^2 + 180t$ kilometres. At $t = 1$ (t measured in hours), a police officer notices they are driving erratically. The motorist claims to have simply suffered a lack of attention: they were in the act of pressing the brakes even as the officer noticed their speed.

At $t = 1$, how fast was the motorist going, and were they pressing the gas or the brake?

Challenge: What about $t = 2$?

Power Rule – Corollary 2.6.17

$\frac{\mathrm{d}}{\mathrm{d}x}\{x^a\} = ax^{a-1}$ (where defined)

Recall that a sphere of radius r has volume $V = \frac{4}{3}\pi r^3$.
Suppose you are winding twine into a gigantic twine ball, filming the process, and trying to make a viral video. You can wrap one cubic meter of twine per hour. (In other words, when we have V cubic meters of twine, we're at time V hours.) How fast is the radius of your spherical twine ball increasing?

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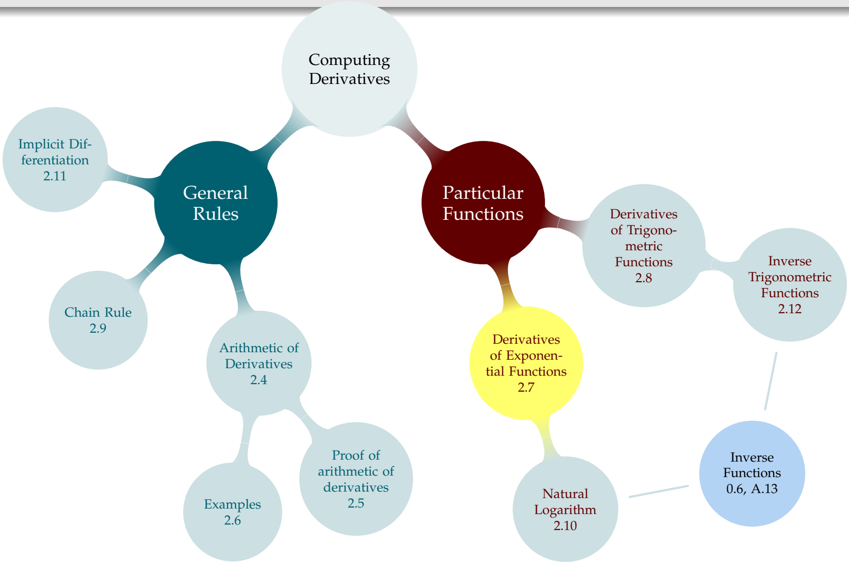
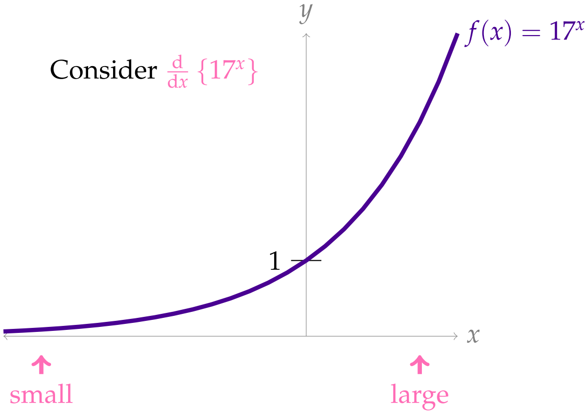


Exponential Functions

Consider $\frac{d}{dx}\{17^x\}$



$f(x)$ is always increasing, so $f'(x)$ is always positive.
$f'(x)$ might look similar to $f(x)$.

Exponential Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} = 17^x \cdot \underbrace{\lim_{h\to 0}\frac{(17^h - 1)}{h}}_{\text{constant}}$$

Given what you know about $\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\}$, **is it possible** that $\displaystyle\lim_{h\to 0}\frac{17^h - 1}{h} = 0$?

A. Sure, there's no reason we've seen that would make it impossible.
B. No, it couldn't be 0, that wouldn't make sense.
C. I do not feel equipped to answer this question.

$$\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} = 17^x \cdot \underbrace{\lim_{h\to 0}\frac{(17^h-1)}{h}}_{\text{constant}}$$

Given what you know about $\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\}$, **is it possible** that $\displaystyle\lim_{h\to 0}\frac{17^h-1}{h} = \infty$?

A. Sure, there's no reason we've seen that would make it impossible.
B. No, it couldn't be ∞, that wouldn't make sense.
C. I do not feel equipped to answer this question.

$$\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} = 17^x \cdot \underbrace{\lim_{h\to 0}\frac{(17^h-1)}{h}}_{\text{constant}}$$

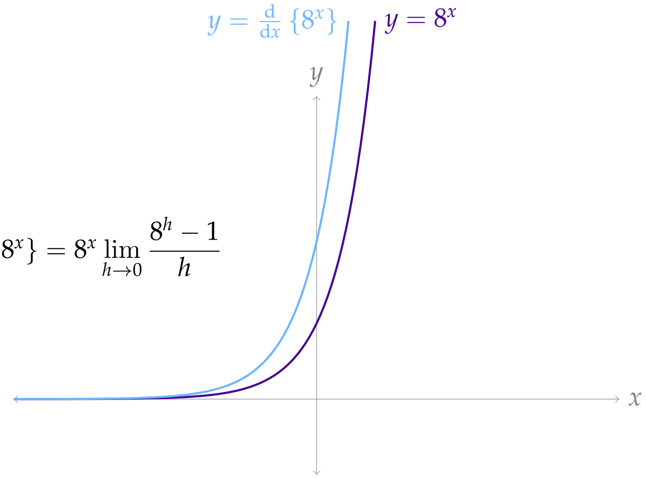
h	$\dfrac{17^h-1}{h}$
0.001	2.83723068608
0.00001	2.83325347992
0.0000001	2.83321374583
0.000000001	2.83321344163

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\{17^x\} &= \lim_{h\to 0}\frac{17^{x+h}-17^x}{h}\\
&= \lim_{h\to 0}\frac{17^x 17^h-17^x}{h}\\
&= \lim_{h\to 0}\frac{17^x(17^h-1)}{h}\\
&= 17^x\lim_{h\to 0}\frac{(17^h-1)}{h}
\end{aligned}$$

In general, for any positive number a,

$$\frac{\mathrm{d}}{\mathrm{d}x}\{a^x\} = a^x\lim_{h\to 0}\frac{a^h-1}{h}$$

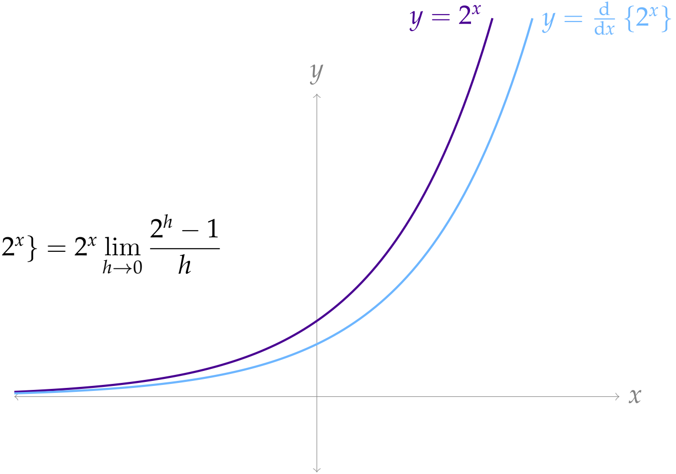
Exponential Functions



$$\frac{d}{dx}\{8^x\} = 8^x \lim_{h\to 0} \frac{8^h - 1}{h}$$

Exponential Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{2^x\right\} = 2^x \lim_{h\to 0} \frac{2^h - 1}{h}$$



In general, for any positive number a, $\frac{\mathrm{d}}{\mathrm{d}x}\{a^x\} = a^x \lim_{h\to 0} \frac{a^h - 1}{h}$

Euler's Number – Theorem 2.7.4

We define e to be the unique number satisfying

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

$e \approx 2.71828182845904523536028747135266 24\ldots$ (Wikipedia)

Theorem 2.7.4 and Corollary 2.10.6

Using this definition of e,

$$\frac{\mathrm{d}}{\mathrm{d}x}\{e^x\} = e^x \underbrace{\lim_{h\to 0}\frac{e^h-1}{h}}_{1} = e^x$$

In general, $\displaystyle\lim_{h\to 0}\frac{a^h-1}{h} = \log_e(a)$, so $\frac{\mathrm{d}}{\mathrm{d}x}\{a^x\} = a^x\log_e(a)$

That $\displaystyle\lim_{h\to 0}\frac{a^h-1}{h} = \log_e(a)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\{a^x\} = a^x\log_e(a)$ are consequences of

$$a^x = \left(e^{\log_e(a)}\right)^x = e^{x\log_e(a)}$$

For the details, see the end of Section 2.7.

Things to Have Memorized

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{e^x\right\} = e^x$$

When a is any constant,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{a^x\right\} = a^x \log_e(a)$$

Let $f(x) = \dfrac{e^x}{3x^5}$. When is the tangent line to $f(x)$ horizontal?

Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\left\{e^{3x}\right\}$

Suppose the deficit, in millions, of a fictitious country is given by

$$f(x) = e^x(4x^3 - 12x^2 + 14x - 4)$$

where x is the number of years since the current leader took office. Suppose the leader has been in power for exactly two years.

1. Is the deficit increasing or decreasing?

2. Is the rate at which the deficit is growing increasing or decreasing?

Included Work

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