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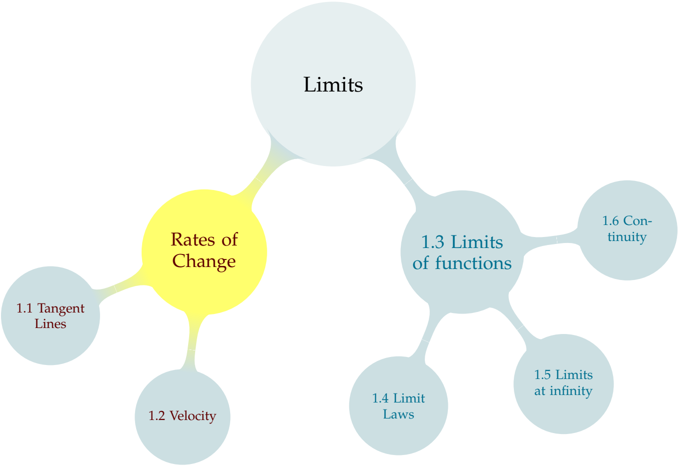
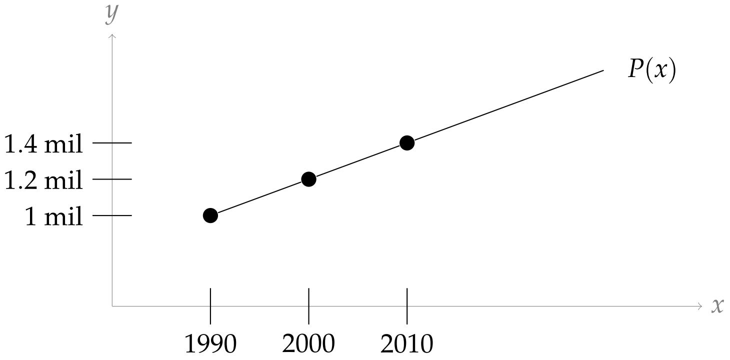


# Rates of Change

Suppose the population of a small country was 1 million individuals in 1990, and is growing at a steady rate of 20,000 individuals per year.

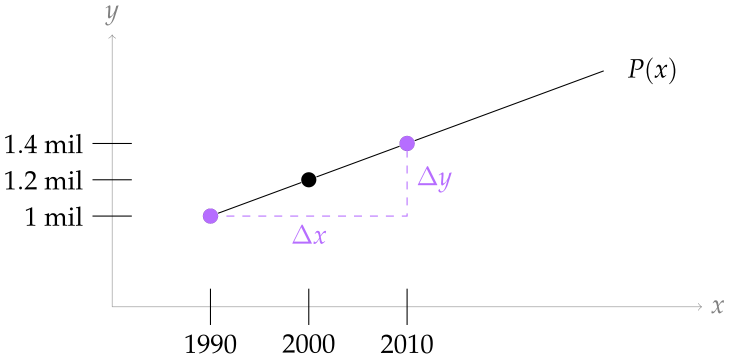


## Definition

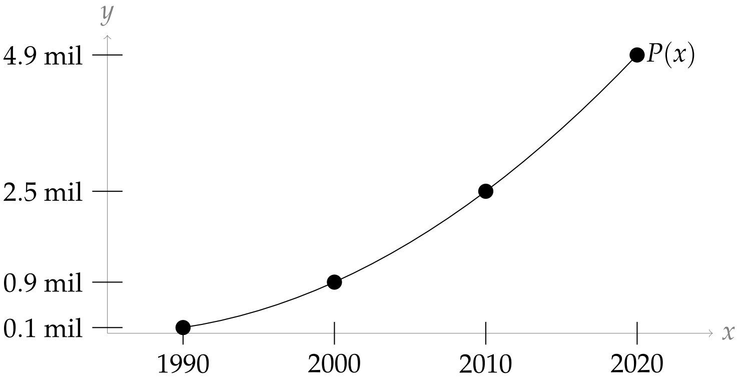
The **slope** of a line that passes through the points $(x_1, y_1)$ and $(x_2, y_2)$ is "rise over run"

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

This is also called the **rate of change** of the function.
If a line has equation $y = mx + b$, its slope is $m$.


y
P(x)
1.4 mil
1.2 mil
1 mil
Δy
Δx
x
1990
2000
2010

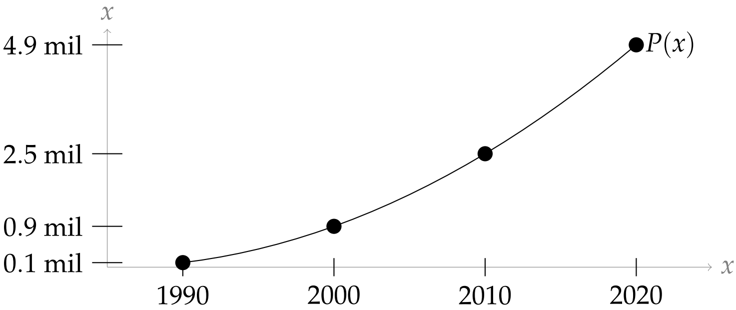
Suppose the population of a small country is given in the chart below.



## Definition

Let $y = f(x)$ be a curve that passes through $(x_1, y_1)$ and $(x_2, y_2)$. Then the **average rate of change** of $f(x)$ when $x_1 \leq x \leq x_2$ is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$


$x$
4.9 mil
2.5 mil
0.9 mil
0.1 mil
$P(x)$
1990
2000
2010
2020
$x$

## Average Rate of Change and Slope

The average rate of change of a function $f(x)$ on the interval $[a, b]$ (where $a \neq b$) is "change in output" divided by "change in input:"

$$\frac{f(b) - f(a)}{b - a}$$

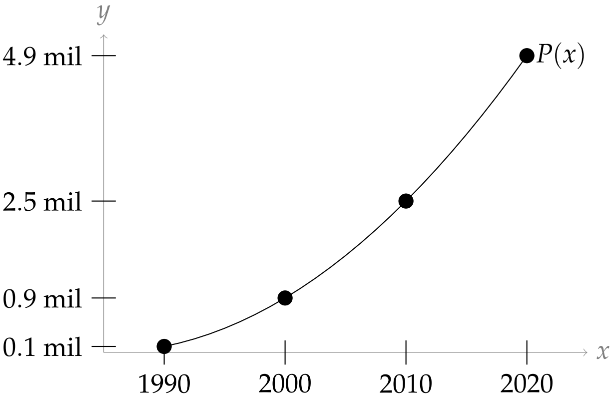
If the function $f(x)$ is a line, then the slope of the line is "rise over run,"

$$\frac{f(b) - f(a)}{b - a}$$

If a function is a line, its slope is the same as its average rate of change, which is the same for every interval.

If a function is not a line, its average rate of change might be different for different intervals, and we don't have a definition (yet) for its "slope."

How fast was this population growing in the year 2010? (What was its instantaneous rate of change?)



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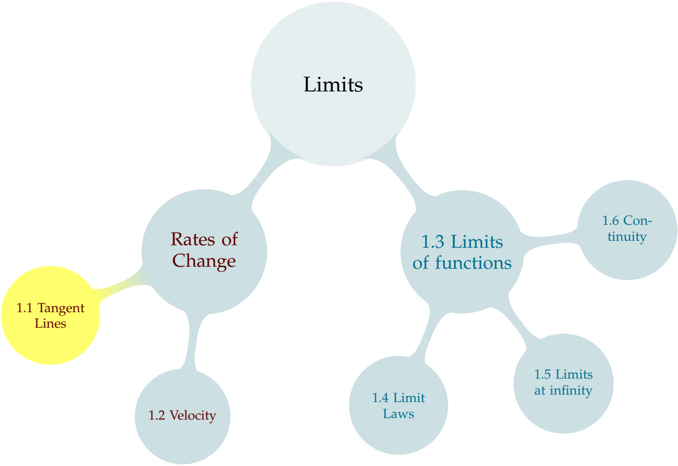


## Definition

The **secant line** to the curve $y = f(x)$ through points $R$ and $Q$ is a line that passes through $R$ and $Q$.

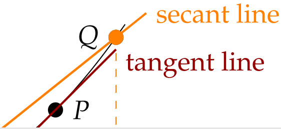
We call the slope of the secant line the **average rate of change of $f(x)$ from $R$ to $Q$.**

## Definition

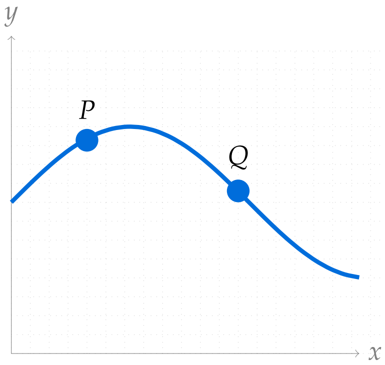
The **tangent line** to the curve $y = f(x)$ at point $P$ is a line that

- passes through $P$ and
- has the same slope as $f(x)$ at $P$.

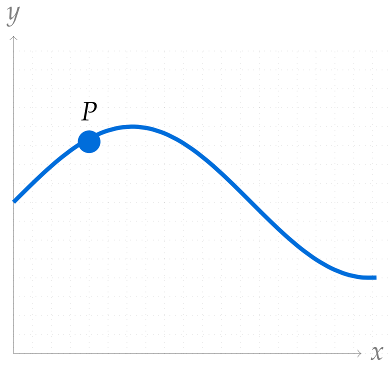
We call the slope of the tangent line the **instantaneous rate of change of $f(x)$ at $P$.**



On the graph below, draw the secant line to the curve through points $P$ and $Q$.




On the graph below, draw the tangent line to the curve at point $P$.



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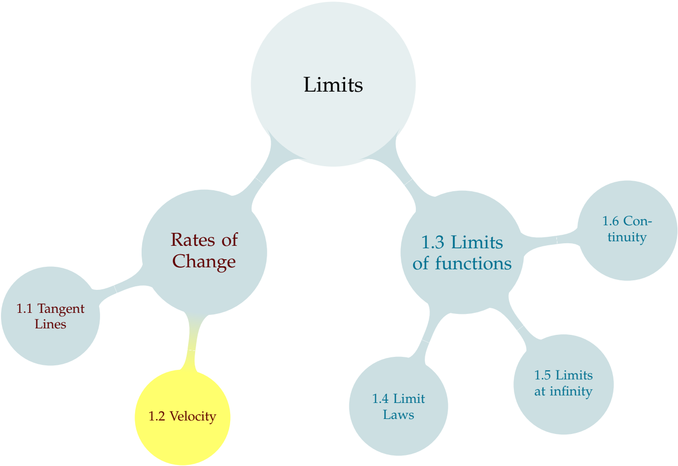



$y = s(t)$
km
6
3
$t$
8 : 00
8 : 30


It took $\frac{1}{2}$ hour to bike 6 km. 12 kph represents the:

A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$

B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$

C. tangent line to $y = s(t)$ at $t = 8:30$

D. slope of the tangent line to $y = s(t)$ at $t = 8:30$


At 8:25, the speedometer on my bike reads 5 kph. 5 kph represents the:

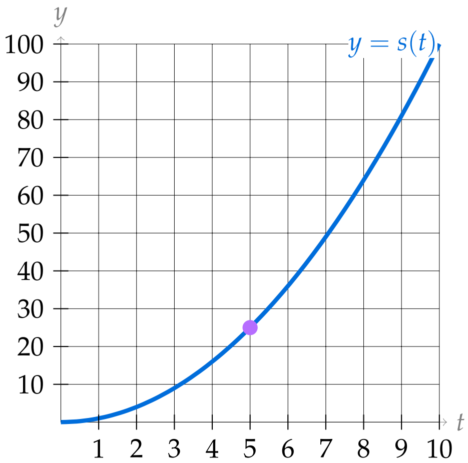
A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$

B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$

C. tangent line to $y = s(t)$ at $t = 8:25$

D. slope of the tangent line to $y = s(t)$ at $t = 8:25$

Suppose the distance from the ground $s$ (in meters) of a helium-filled balloon at time $t$ over a 10-second interval is given by $s(t) = t^2$. Try to estimate how fast the balloon is rising when $t = 5$.



Let's look for an algebraic way of determining the velocity of the balloon when $t = 5$.

# Our First Limit

Average Velocity, $t = 5$ to $t = 5 + h$:

$$\begin{aligned}\frac{\Delta s}{\Delta t} &= \frac{s(5+h) - s(5)}{h} \\ &= \frac{(5+h)^2 - 5^2}{h} \\ &= 10 + h \qquad \text{when } h \neq 0\end{aligned}$$

When $h$ is very small,

$$\text{Vel} \approx 10$$

# Limit Notation

We write:

$$\lim_{h \to 0}(10 + h) = 10$$

We say: "The limit as $h$ goes to 0 of $(10 + h)$ is 10."

It means: As $h$ gets extremely close to 0, $(10 + h)$ gets extremely close to 10.

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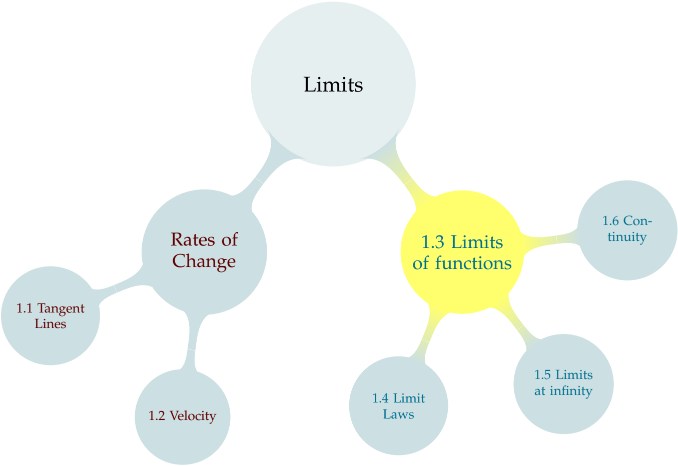


## Notation 1.3.1 and Definition 1.3.3

$$\lim_{x \to a} f(x) = L$$

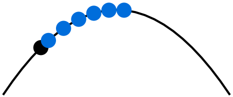
where $a$ and $L$ are real numbers

We read the above as "the limit as $x$ goes to $a$ of $f(x)$ is $L$."

Its meaning is: as $x$ gets very close to (but not equal to) $a$, $f(x)$ gets very close to $L$.

# Finding Slopes of Tangent Lines

We NEED limits to find slopes of tangent lines.



Slope of secant line: $\frac{\Delta y}{\Delta x}$, $\Delta x \neq 0$.
Slope of tangent line: can't do the same way.

If the position of an object at time $t$ is given by $s(t)$, then its instantaneous velocity is given by

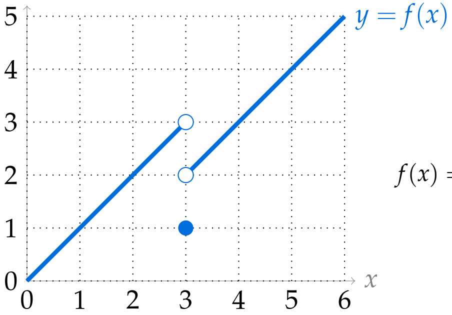
$$\lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

# Evaluating Limits

$$\text{Let } f(x) = \frac{x^3 + x^2 - x - 1}{x - 1}.$$

We want to evaluate $\lim_{x \to 1} f(x)$.

# One-Sided Limits




$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x - 1 & \text{if } x > 3 \end{cases}$$

What do you think $\lim\limits_{x \to 3} f(x)$ should be?

## Definition 1.3.7

The limit as $x$ goes to $a$ **from the left** of $f(x)$ is written

$$\lim_{x \to a^-} f(x)$$

We only consider values of $x$ that are less than $a$.

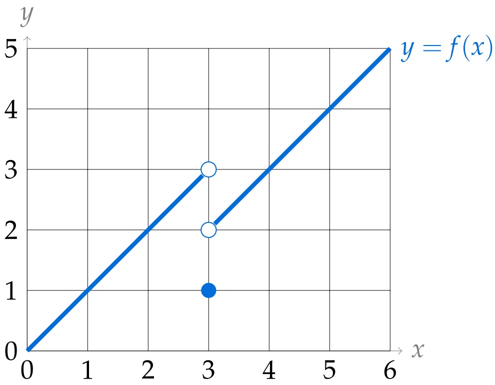
The limit as $x$ goes to $a$ **from the right** of $f(x)$ is written

$$\lim_{x \to a^+} f(x)$$

We only consider values of $x$ greater than $a$.

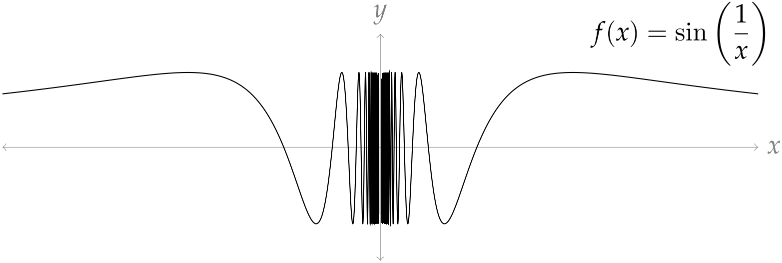
## Theorem 1.3.8

In order for $\lim_{x \to a} f(x)$ to exist, both one-sided limits must exist and be equal.



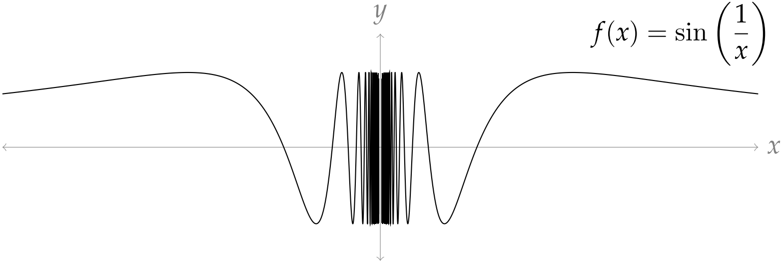
Consider the function $f(x) = \dfrac{1}{(x-1)^2}$. For what value(s) of $x$ is $f(x)$ not defined?

# A Stranger Limit Example



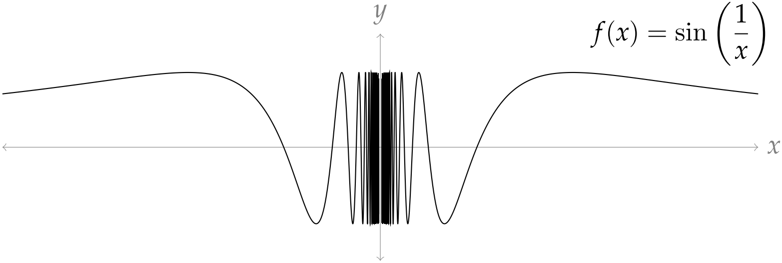

What is $\lim_{x\to\infty} f(x)$ ?

# A Stranger Limit Example



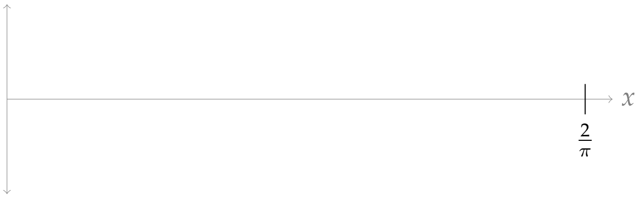

What is $\lim_{x \to 0} f(x)$ ?

# A Stranger Limit Example

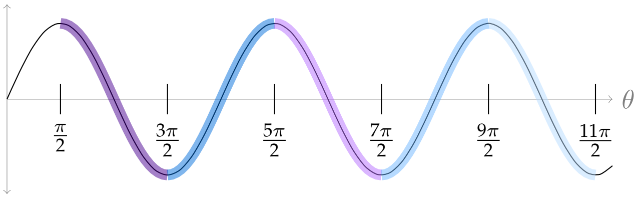



What is $\lim_{x \to \pi} f(x)$ ?

# Optional: Sketching $f(x) = \sin\left(\frac{1}{x}\right)$



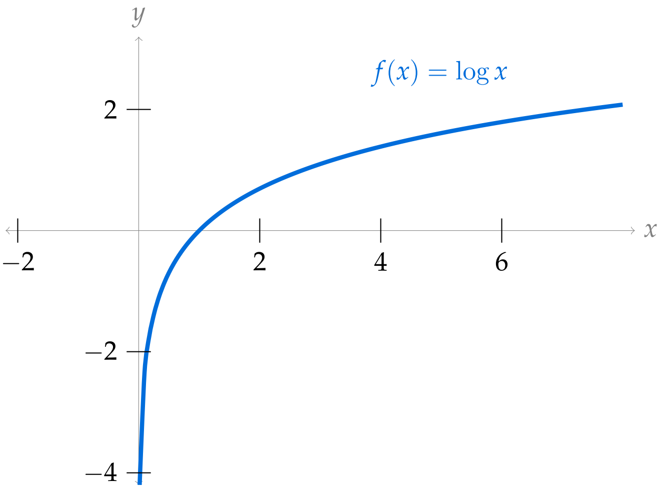

$y = \sin\frac{1}{x}$




$y = \sin\theta$

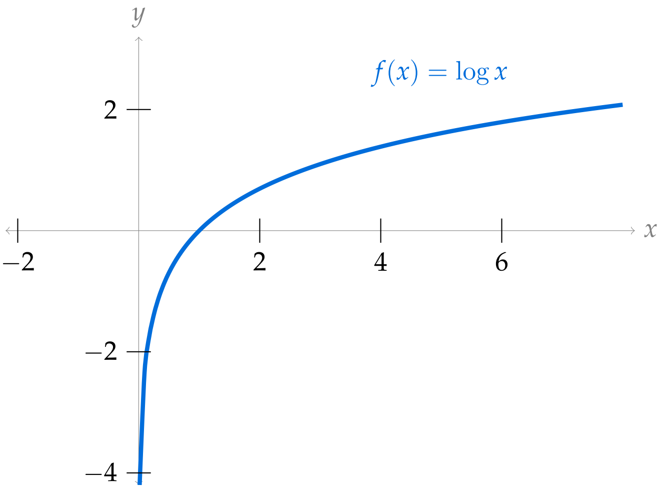
# Limits and the Natural Logarithm

Where is $f(x)$ defined, and where is it not defined?

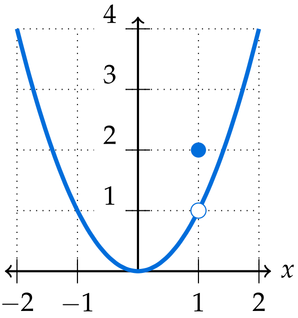


# Limits and the Natural Logarithm

What can you say about the limit of $f(x)$ near 0?



$$f(x) = \begin{cases} x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}$$




What is $\lim_{x\to 1} f(x)$?

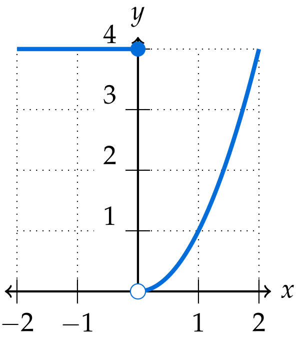
A. $\lim_{x\to 1} f(x) = 2$

B. $\lim_{x\to 1} f(x) = 1$

C. $\lim_{x\to 1} f(x)$ DNE

D. none of the above

$$f(x) = \begin{cases} 4 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$




What is $\lim\limits_{x \to 0} f(x)$? What is $\lim\limits_{x \to 0^+} f(x)$? What is $f(0)$?

A. $\lim\limits_{x \to 0^+} f(x) = 4$

B. $\lim\limits_{x \to 0^+} f(x) = 0$

C. $\lim\limits_{x \to 0^+} f(x) = \begin{cases} 4 & x \leq 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

Suppose $\lim_{x\to 3^-} f(x) = 1$ and $\lim_{x\to 3^+} f(x) = 1.5$.

Does $\lim_{x\to 3} f(x)$ exist?

A. Yes, certainly, because the limits from both sides exist.

B. No, never, because the limit from the left is not the same as the limit from the right.

C. Can't tell. For some functions is might exist, for others not.

Suppose $\lim\limits_{x\to 3^-} f(x) = 22 = \lim\limits_{x\to 3^+} f(x)$.

Does $\lim\limits_{x\to 3} f(x)$ exist?

A. Yes, certainly, because the limits from both sides exist and are equal to each other.

B. No, never, because we only talk about one-sided limits when the actual limit doesn't exist.

C. Can't tell. We need to know the value of the function at $x = 3$.

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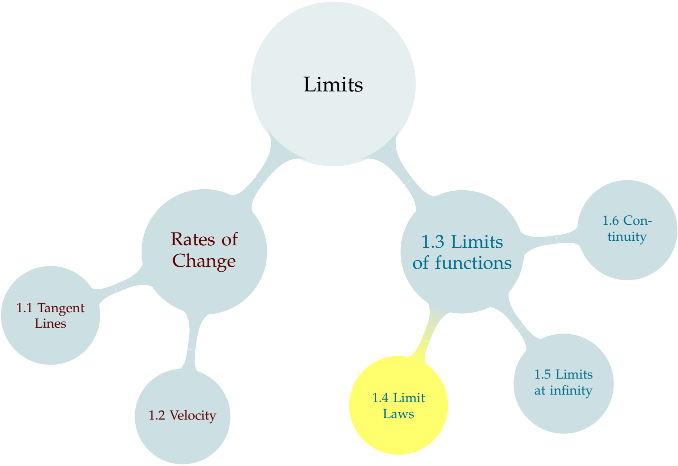


# Calculating Limits in Simple Situations

## Direct Substitution – Theorem 1.4.10

If $f(x)$ is a polynomial or rational function, and $a$ is in the domain of $f$, then:

$$\lim_{x\to a} f(x) = f(a).$$

Calculate: $\displaystyle\lim_{x\to 3}\left(\frac{x^2-9}{x+3}\right)$

Calculate: $\displaystyle\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)$

## Algebra with Limits: Theorem 1.4.2

Suppose $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$, where $F$ and $G$ are both real numbers. Then:

- $\lim_{x\to a}(f(x)+g(x)) = F+G$
- $\lim_{x\to a}(f(x)-g(x)) = F-G$
- $\lim_{x\to a}(f(x)g(x)) = FG$
- $\lim_{x\to a}(f(x)/g(x)) = F/G$ provided $G \neq 0$

Calculate: $\lim_{x\to 1}\left[\frac{2x+4}{x+2} + 13\left(\frac{x+5}{3x}\right)\left(\frac{x^2}{2x-1}\right)\right]$

# LIMITS INVOLVING POWERS AND ROOTS

Which of the following gives a real number?

A. $4^{\frac{1}{2}}$ B. $(-4)^{\frac{1}{2}}$ C. $4^{-\frac{1}{2}}$ D. $(-4)^{-\frac{1}{2}}$

E. $8^{1/3}$ F. $(-8)^{1/3}$ G. $8^{-1/3}$ H. $(-8)^{-1/3}$

## Powers of Limits – Theorem 1.4.8

If $n$ is a positive integer, and $\lim_{x\to a} f(x) = F$ (where $F$ is a real number), then:

$$\lim_{x\to a} (f(x))^n = F^n.$$

Furthermore, unless $n$ is even and $F$ is negative,

$$\lim_{x\to a} (f(x))^{1/n} = F^{1/n}$$

$$\lim_{x\to 4} (x+5)^{1/2}$$

# Cautionary Tales

- $$\lim_{x\to 0} \frac{(5+x)^2-25}{x}$$
- $$\lim_{x\to 3} \left(\frac{x-6}{3}\right)^{1/8}$$
- $$\lim_{x\to 0} \frac{32}{x}$$
- $$\lim_{x\to 5} \left(x^2+2\right)^{1/3}$$

Suppose you want to evaluate $\lim_{x\to 1} f(x)$, but $f(1)$ doesn't exist. What does that tell you?

A $\lim_{x\to 1} f(x)$ may exist, and it may not exist.

B We can find $\lim_{x\to 1} f(x)$ by plugging in 1 to $f(x)$.

C Since $f(1)$ doesn't exist, it is not meaningful to talk about $\lim_{x\to 1} f(x)$.

D Since $f(1)$ doesn't exist, automatically we know $\lim_{x\to 1} f(x)$ does not exist.

E $\lim_{x\to 1} f(x)$ does not exist if we are "dividing by zero," but may exist otherwise.

Which of the following statements is true about $\lim_{x\to 0} \dfrac{\sin x}{x^3 - x^2 + x}$?

A $\lim_{x\to 0} \dfrac{\sin x}{x^3 - x^2 + x} = \dfrac{\sin 0}{0^3 - 0^2 + 0} = \dfrac{0}{0}$

B Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not rational, its limit at 0 does not exist.

C Since the numerator and denominator of $\dfrac{\sin x}{x^3 - x^2 + x}$ are both 0 when $x = 0$, the limit exists.

D Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not defined at 0, plugging in $x = 0$ will not tell us the limit.

E Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ consists of the quotient of polynomials and trigonometric functions, its limit exists everywhere.

Which of the following statements is true about $\displaystyle\lim_{x\to 1} \frac{\sin x}{x^3 - x^2 + x}$?

A $\displaystyle\lim_{x\to 1} \frac{\sin x}{x^3 - x^2 + x} = \frac{\sin 1}{1^3 - 1^2 + 1} = \sin 1$

B Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not rational, its limit at 1 does not exist.

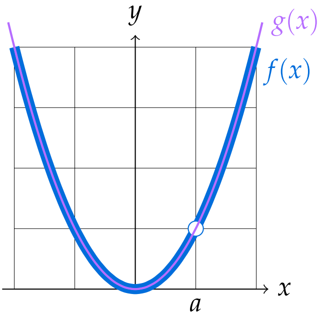
C Since the function $\dfrac{\sin x}{x^3 - x^2 + x}$ is not defined at 1, plugging in $x = 1$ will not tell us the limit.

D Since the numerator and denominator of $\dfrac{\sin x}{x^3 - x^2 + x}$ are both 0 when $x = 1$, the limit exists.

## Functions that Differ at a Single Point – Theorem 1.4.12

Suppose $\lim_{x \to a} g(x)$ exists, and $f(x) = g(x)$ when $x$ is close to $a$ (but not necessarily equal to $a$).

Then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.



Evaluate $\lim_{x\to 1} \dfrac{x^3+x^2-x-1}{x-1}$.

Evaluate $\lim\limits_{x \to 5} \dfrac{\sqrt{x+20} - \sqrt{4x+5}}{x-5}$

# A Few Strategies for Calculating Limits

First, hope that you can directly substitute (plug in). If your function is made up of the sum, difference, product, quotient, or power of polynomials, you can do this provided the function exists where you're taking the limit.

$$\lim_{x\to 1}\left(\sqrt{35+x^5}+\frac{x-3}{x^2}\right)^3 =$$

To take a limit outside the domain of a function (that is made up of the sum, difference, product, quotient, or power of polynomials) try to simplify and cancel.

$$\lim_{x\to 0}\frac{x+7}{\frac{1}{x}-\frac{1}{2x}}$$

Otherwise, you can try graphing the function, or making a table of values, to get a better picture of what is going on.

# Denominators Approaching Zero

$$\lim_{x\to1} \frac{1}{(x-1)^2}$$

$$\lim_{x\to1} \frac{-1}{(x-1)^2}$$

$$\lim_{x\to1^-} \frac{1}{x-1}$$

$$\lim_{x\to1^+} \frac{1}{x-1}$$

# Denominators Approaching Zero

Now You

$$\lim_{x\to 2^+} \frac{x}{x^2-4}$$

$$\lim_{x\to 2^-} \frac{x}{4-x^2}$$

$$\lim_{x\to 2} \frac{x-2}{x^2-4}$$

## Squeeze Theorem – Theorem 1.4.17

Suppose, when $x$ is near (but not necessarily equal to) $a$, we have functions $f(x)$, $g(x)$, and $h(x)$ so that

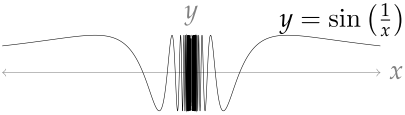
$$f(x) \leq g(x) \leq h(x)$$

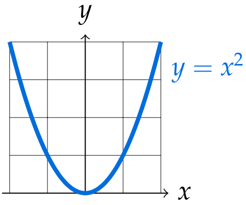
and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$. Then $\lim_{x \to a} g(x) = \lim_{x \to a} f(x)$.

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

Evaluate:

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$



$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

# Included Work

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