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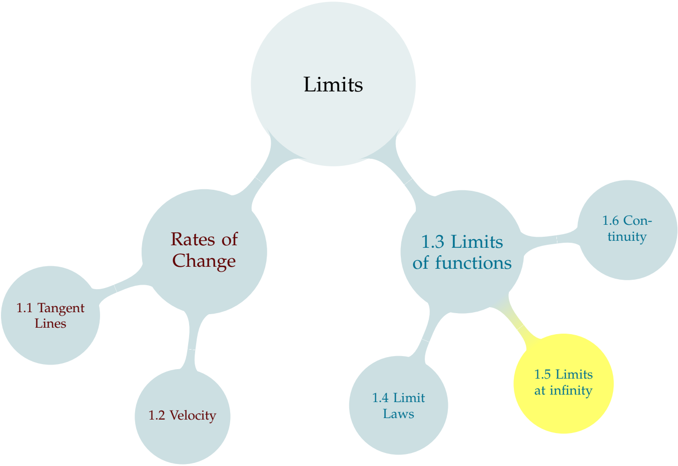


End Behavior

We write:

$$\lim_{x \to \infty} f(x) = L$$

to express that, as x grows larger and larger, $f(x)$ approaches L.

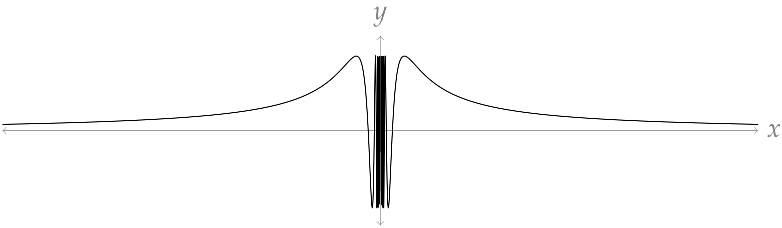
Similarly, we write:

$$\lim_{x \to -\infty} f(x) = L$$

to express that, as x grows more and more strongly negative, $f(x)$ approaches L.

If L is a number, we call $y = L$ a horizontal asymptote of $f(x)$.

Horizontal Asymptotes



$y = 0$ is a horizontal asymptote for $y = \sin\left(\frac{1}{x}\right)$

Common Limits at Infinity

$$\lim_{x\to\infty} 13 =$$
$$\lim_{x\to-\infty} 13 =$$

$$\lim_{x\to\infty} \frac{1}{x} =$$
$$\lim_{x\to-\infty} \frac{1}{x} =$$

$$\lim_{x\to\infty} x^2 =$$
$$\lim_{x\to-\infty} x^2 =$$

$$\lim_{x\to\infty} x^3 =$$
$$\lim_{x\to-\infty} x^3 =$$

$$\lim_{x\to-\infty} x^{5/3} =$$
$$\lim_{x\to-\infty} x^{2/3} =$$

Arithmetic with Limits at Infinity

Now You

$$\lim_{x\to\infty}\left(x+\frac{x^2}{10}\right)=$$

$$\lim_{x\to\infty}\left(x-\frac{x^2}{10}\right)=$$

$$\lim_{x\to-\infty}\left(x^2+x^3+x^4\right)=$$

$$\lim_{x\to-\infty}(x+13)\left(x^2+13\right)^{1/3}=$$

Calculating Limits at Infinity

$$\lim_{x\to\infty} \frac{x^2+2x+1}{x^3}$$

Calculating Limits at Infinity

$$\lim_{x\to-\infty}\left(x^{7/3}-x^{5/3}\right)$$

Again: factor out largest power of x.

Calculating Limits at Infinity

Suppose the height of a bouncing ball is given by $h(t) = \frac{\sin(t)+1}{t}$, for $t \geq 1$. What happens to the height over a long period of time?

Calculating Limits at Infinity

Now You

$$\lim_{x\to\infty} \sqrt{x^4+x^2+1}-\sqrt{x^4+3x^2}$$

Now You

Evaluate $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{3+x^2}}{3x}$

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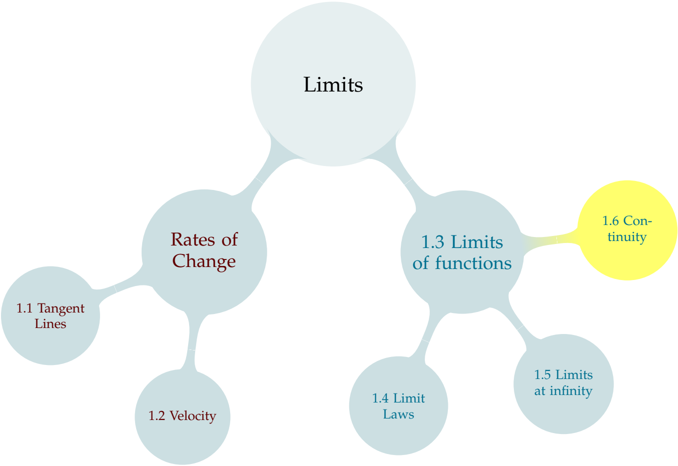
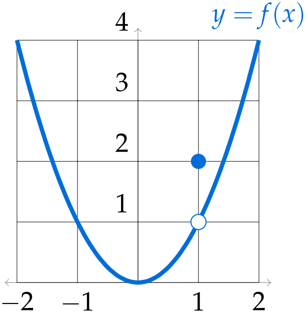


Continuity

Definition 1.6.1

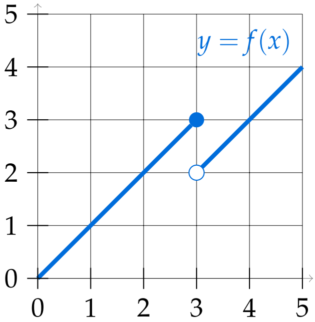
A function $f(x)$ is continuous at a point a if $\lim_{x\to a} f(x)$ exists AND is equal to $f(a)$.



Does $f(x)$ exist at $x = 1$?
Is $f(x)$ continuous at $x = 1$?

Definitions 1.6.1 and 1.6.2

A function $f(x)$ is continuous from the left at a point a if $\lim\limits_{x \to a^-} f(x)$ exists AND is equal to $f(a)$.



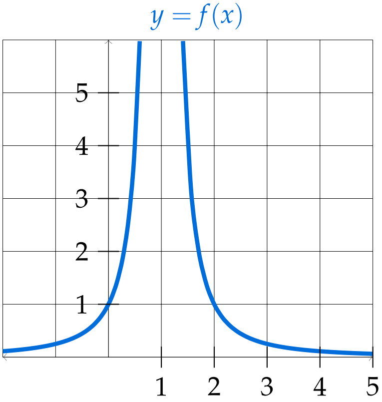
Is $f(x)$ continuous at $x = 3$?

Is $f(x)$ continuous from the left at $x = 3$?

Is $f(x)$ continuous from the right at $x = 3$?

Definition

A function $f(x)$ is continuous at a point a if $\lim_{x \to a} f(x)$ exists AND is equal to $f(a)$.



Definition

A function $f(x)$ is continuous at a point a if $\lim_{x\to a} f(x)$ exists AND is equal to $f(a)$.

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Is $f(x)$ continuous at 0?

Continuous Functions

Functions made by adding, subtracting, multiplying, dividing, and taking appropriate powers of polynomials are continuous for every point in their domain.

$$f(x) = \frac{x^2}{2x-10} - \left(\frac{x^2+2x-1}{x-1} + \frac{\sqrt[5]{25-x} - \frac{1}{x}}{x+2} \right)^{1/3}$$

A continuous function is continuous for every point in $\mathbb{R}$.

We say $f(x)$ is continuous over (a, b) if it is continuous at every point in (a, b).

Common Functions – Theorem 1.6.8

Functions of the following types are continuous over their domains:

- polynomials and rationals
- roots and powers
- trig functions and their inverses
- exponential and logarithm
- The products, sums, differences, quotients, powers, and compositions of continuous functions

Where is the following function continuous?

$$f(x) = \left(\frac{\sin x}{(x-2)(x+3)} + e^{\sqrt{x}} \right)^3$$

A Technical Point

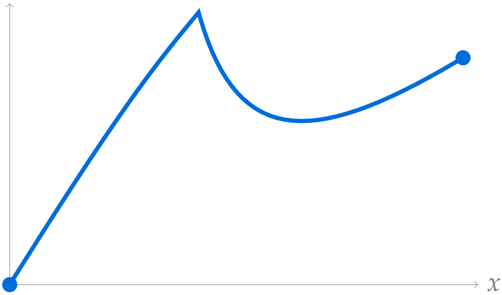
Definition 1.6.3

A function $f(x)$ is continuous on the closed interval $[a,b]$ if:

- $f(x)$ is continuous over (a,b), and
- $f(x)$ is continuous from the left at b, and
- $f(x)$ is continuous from the right at a

Intermediate Value Theorem (IVT) – Theorem 1.6.12

Let $a < b$ and let $f(x)$ be continuous over $[a, b]$. If y is any number between $f(a)$ and $f(b)$, then there exists c in (a, b) such that $f(c) = y$.



Intermediate Value Theorem (IVT) – Theorem 1.6.12

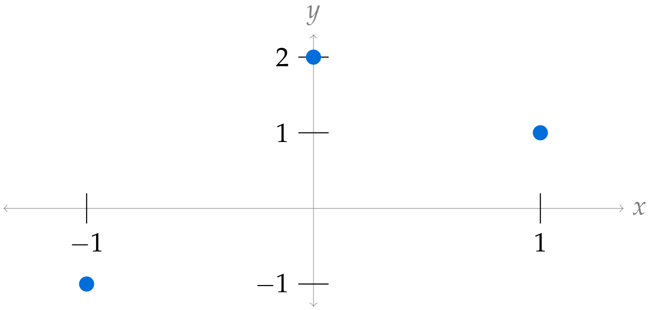
Let $a < b$ and let $f(x)$ be continuous over $[a, b]$. If y is any number between $f(a)$ and $f(b)$, then there exists c in (a, b) such that $f(c) = y$.

Suppose your favourite number is 45.54. At noon, your car is parked, and at 1pm you're driving 100kph.

USING IVT TO FIND ROOTS: "BISECTION METHOD"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value x for which $f(x) = 0$. Let's find some points:

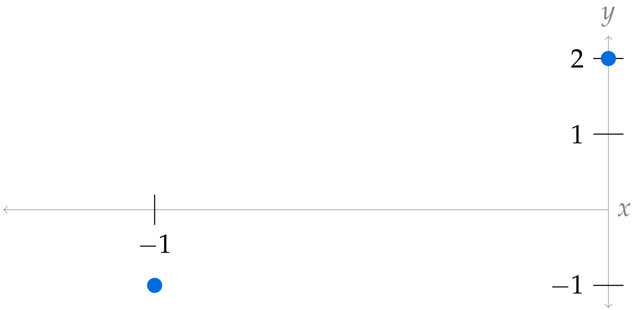
$f(0) = 2$ $\qquad$ $f(1) = 1$ $\qquad$ $f(-1) = -1$



Using IVT to Find Roots: "Bisection Method"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value x for which $f(x) = 0$.

$f(0) = 2, f(-1) = -1$



Use the Intermediate Value Theorem to show that there exists some solution to the equation $\ln x \cdot e^x = 4$, and give a reasonable interval where that solution might occur.

Now You

Use the Intermediate Value Theorem to give a reasonable interval where the following is true: $e^x = \sin(x)$. (Don't use a calculator – use numbers you can easily evaluate.)

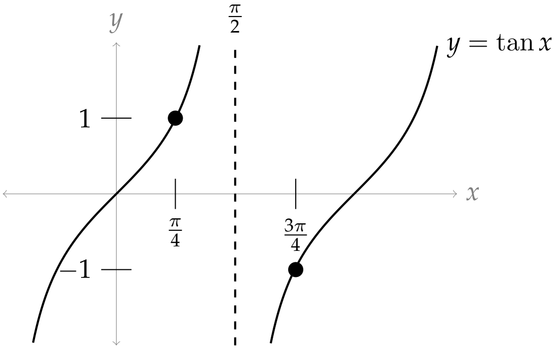
Now You

Is there any value of x so that $\sin x = \cos(2x) + \frac{1}{4}$?

Now You

Is the following reasoning correct?

- $f(x) = \tan x$ is continuous over its domain, because it is a trigonometric function.
- In particular, $f(x)$ is continuous over the interval $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.
- $f\left(\frac{\pi}{4}\right) = 1$, and $f\left(\frac{3\pi}{4}\right) = -1$.
- Since $f\left(\frac{3\pi}{4}\right) < 0 < f\left(\frac{\pi}{4}\right)$, by the Intermediate Value Theorem, there exists some number c in the interval $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ such that $f(c) = 0$.



y
$\frac{\pi}{2}$
$y = \tan x$
1
-1
$\frac{\pi}{4}$
$\frac{3\pi}{4}$
x

CONTINUITY

Section 1.6 Review

Suppose $f(x)$ is continuous at $x = 1$. Does $f(x)$ have to be defined at $x = 1$?

Suppose $f(x)$ is continuous at $x = 1$ and $\lim_{x\to 1^-} f(x) = 30$.

True or false: $\lim_{x\to 1^+} f(x) = 30$.

Suppose $f(x)$ is continuous at $x = 1$ and $f(1) = 22$. What is $\lim_{x \to 1} f(x)$?

Suppose $\lim_{x \to 1} f(x) = 2$. Must it be true that $f(1) = 2$?

$$f(x) = \begin{cases} ax^2 & x \geq 1 \\ 3x & x < 1 \end{cases}$$

For which value(s) of a is $f(x)$ continuous?

$$f(x) = \begin{cases} \frac{\sqrt{3}x+3}{x^2-3} & x \neq \pm\sqrt{3} \\ a & x = \pm\sqrt{3} \end{cases}$$

For which value(s) of a is $f(x)$ continuous at $x = -\sqrt{3}$?

$$f(x) = \begin{cases} \frac{\sqrt{3}x+3}{x^2-3} & x \neq \pm\sqrt{3} \\ a & x = \pm\sqrt{3} \end{cases}$$

For which value(s) of a is $f(x)$ continuous at $x = \sqrt{3}$?

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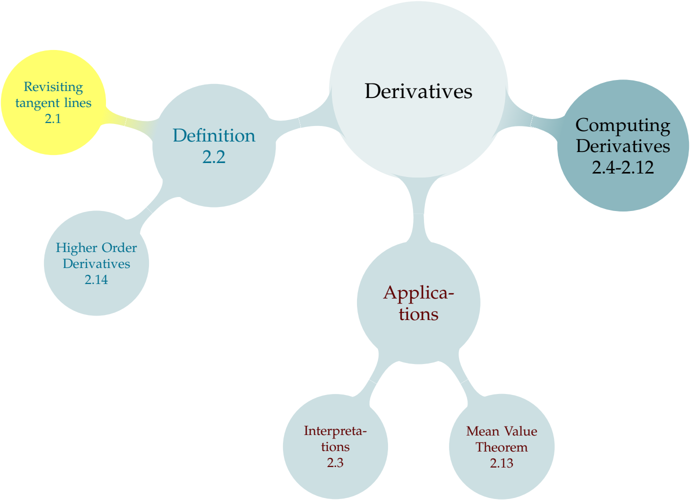
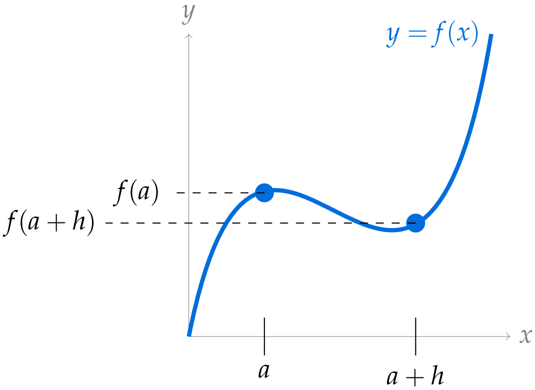


Slope of Secant and Tangent Line

Slope

Recall, the slope of a line is given by any of the following:

$$\frac{\text{rise}}{\text{run}} \qquad \frac{\Delta y}{\Delta x} \qquad \frac{y_2 - y_1}{x_2 - x_1}$$



Slope of secant line: $\frac{f(a+h)-f(a)}{h}$

Slope of tangent line: $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$

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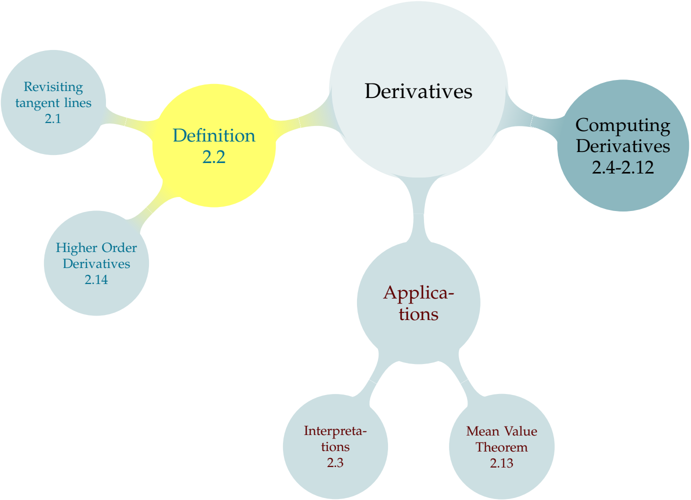


Derivative at a Point

Definition 2.2.1

Given a function $f(x)$ and a point a, the slope of the tangent line to $f(x)$ at a is the derivative of f at a, written $f'(a)$.

So, $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$.

$f'(a)$ is also the instantaneous rate of change of f at a.

Derivative

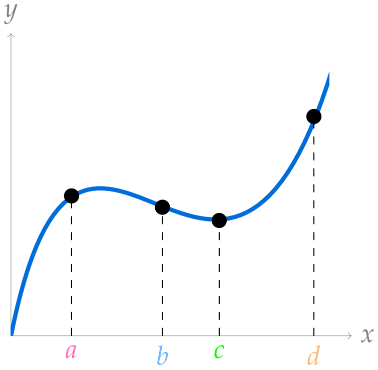
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If $f'(a) > 0$, then f is increasing at a. Its graph "points up."

If $f'(a) < 0$, then f is decreasing at a. Its graph "points down."

If $f'(a) = 0$, then f looks constant or flat at a.

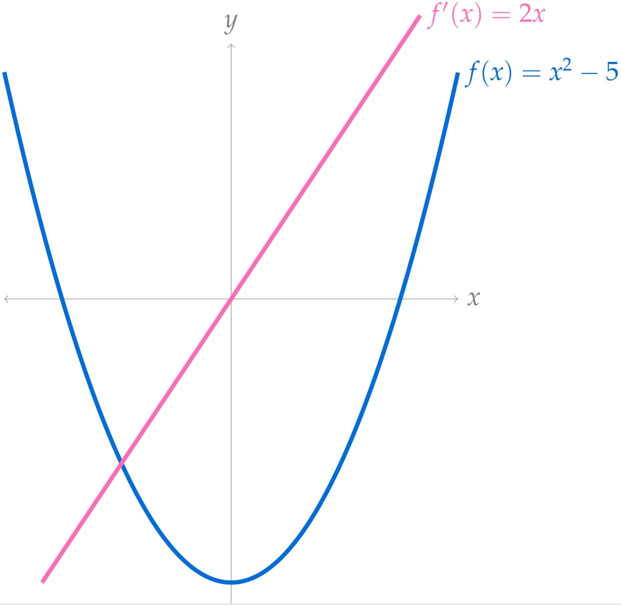
Practice: Increasing and Decreasing



Where is $f'(x) < 0$? Where is $f'(x) > 0$? Where is $f'(x) \approx 0$?

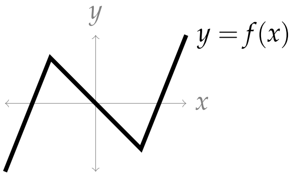
Use the definition of the derivative to find the slope of the tangent line to $f(x) = x^2 - 5$ at the point $x = 3$.

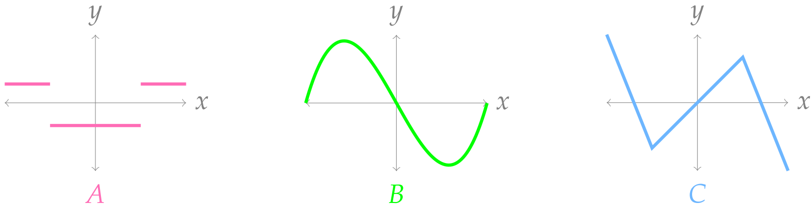
Let's keep the function $f(x) = x^2 - 5$. We just showed $f'(3) = 6$. We can also find its derivative at an arbitrary point x:


y
x
$f'(x) = 2x$
$f(x) = x^2 - 5$

Increasing and Decreasing

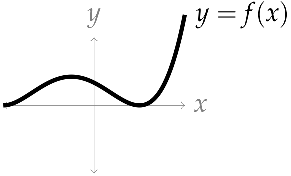
In black is the curve $y = f(x)$. Which of the coloured curves corresponds to $y = f'(x)$?

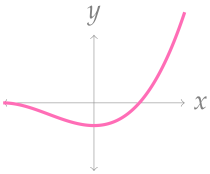




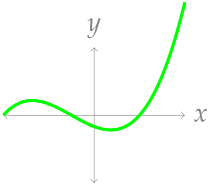
Increasing and Decreasing

In black is the curve $y = f(x)$. Which of the coloured curves corresponds to $y = f'(x)$?

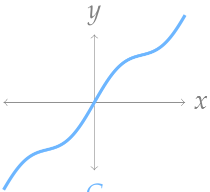




A



B



C

Derivative as a Function – Definition 2.2.6

Let $f(x)$ be a function.
The derivative of $f(x)$ with respect to x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. Notice that x will be a part of your final expression: this is a function.

If $f'(x)$ exists for all x in an interval (a, b), we say that f is differentiable on (a, b).

Notation 2.2.8

The "prime" notation $f'(x)$ and $f'(a)$ is sometimes called Newtonian notation. We will also use Leibnitz notation:

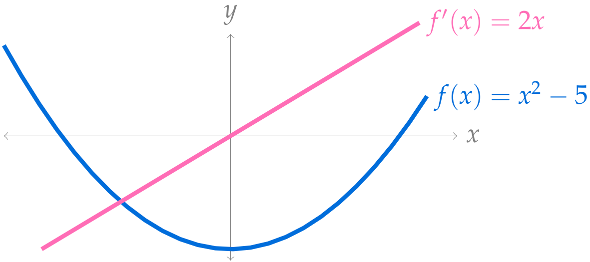
$\dfrac{\mathrm{d}f}{\mathrm{d}x}$	$\dfrac{\mathrm{d}f}{\mathrm{d}x}(a)$	$\dfrac{\mathrm{d}}{\mathrm{d}x}f(x)$	$\dfrac{\mathrm{d}}{\mathrm{d}x}f(x)\Big\vert_{x=a}$
function	number	function	number

Newtonian Notation:

$$f(x) = x^2 + 5 \qquad f'(x) = 2x \qquad f'(3) = 6$$

Leibnitz Notation:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(3) = \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x) = \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x)\Big|_{x=3} =$$



Alternate Definition – Definition 2.2.1

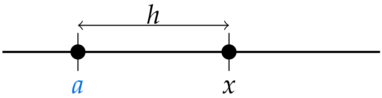
Calculating

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

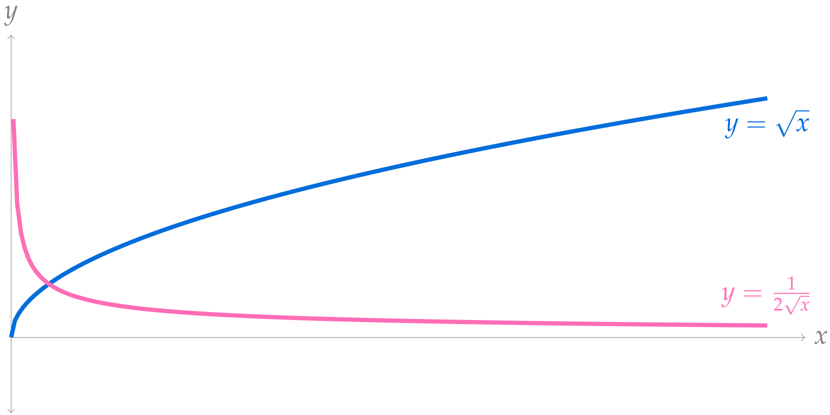
is the same as calculating

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Notice in these scenarios, $h = x - a$.



Let $f(x) = \sqrt{x}$. Using the definition of a derivative, calculate $f'(x)$.



$\boxed{\text{Review:}}$

$$\lim_{x\to\infty} \sqrt{x} = \qquad \lim_{x\to\infty} \frac{1}{2\sqrt{x}} =$$

$$\lim_{x\to 0^+} \sqrt{x} = \qquad \lim_{x\to 0^+} \frac{1}{2\sqrt{x}} =$$

Now You

Using the definition of the derivative, calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{x}\right\}.$$

Using the definition of the derivative, calculate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{2x}{x+1}\right\}$.

Using the definition of the derivative, calculate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{1}{\sqrt{x^2+x}}\right\}$.

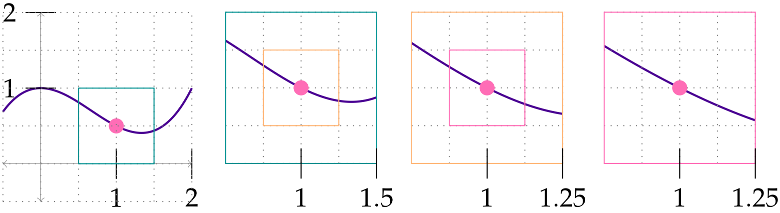
Memorize

The derivative of a function f at a point a is given by the following limit, if it exists:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Zooming In

For a smooth function, if we zoom in at a point, we see a line:



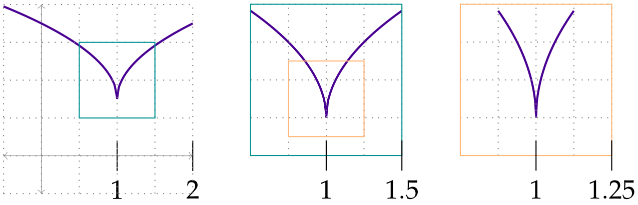
In this example, the slope of our zoomed-in line looks to be about:

$$\frac{\Delta y}{\Delta x} \approx -\frac{1}{2}$$

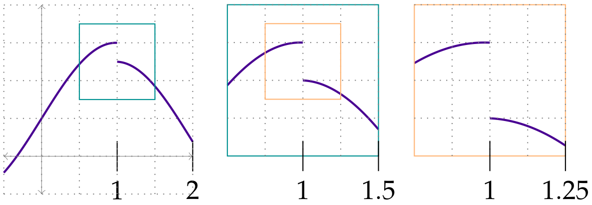
Zooming in on functions that aren't smooth

For a function with a cusp or a discontinuity, even though we zoom in very closely, we don't see simply a single straight line.

Cusp:



Discontinuity:



Alternate Definition – Definition 2.2.1

Calculating

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

is the same as calculating

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Notice in these scenarios, $h = x - a$.

The derivative of $f(x)$ does not exist at $x = a$ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

does not exist.
Note this is the slope of the tangent line to $y = f(x)$ at $x = a$, $\frac{\Delta y}{\Delta x}$.

When Derivatives Don't Exist

What happens if we try to calculate a derivative where none exists?

Find the derivative of $f(x) = x^{1/3}$ at $x = 0$.

Theorem 2.2.14

If the function $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.

Proof:

Let $f(x)$ be a function and let a be a constant in its domain. Draw a picture of each scenario, or say that it is impossible.

$f(x)$ continuous at $x = a$ $f(x)$ differentiable at $x = a$	$f(x)$ continuous at $x = a$ $f(x)$ differentiable at $x = a$
$f(x)$ continuous at $x = a$ $f(x)$ differentiable at $x = a$	$f(x)$ continuous at $x = a$ $f(x)$ differentiable at $x = a$

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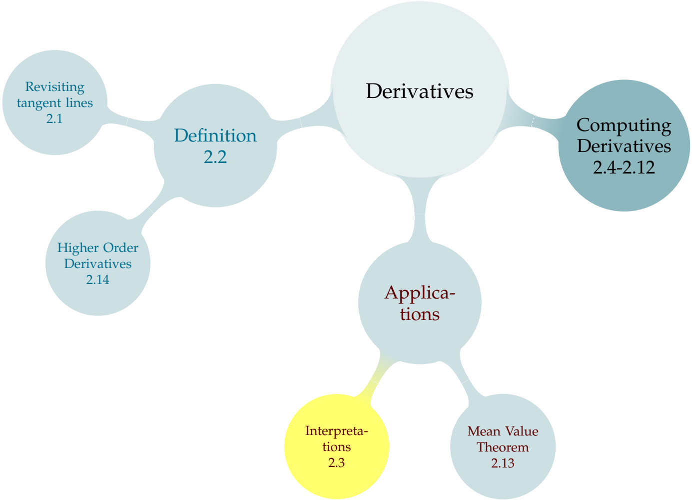

Interpreting the Derivative

The derivative of $f(x)$ at a, written $f'(a)$, is the instantaneous rate of change of $f(x)$ when $x = a$.

Interpreting the Derivative

The derivative of $f(x)$ at a, written $f'(a)$, is the instantaneous rate of change of $f(x)$ when $x = a$.

Suppose $P(t)$ gives the number of people in the world at t minutes past midnight, January 1, 2012. Suppose further that $P'(0) = 156$. How do you interpret $P'(0) = 156$?

Interpreting the Derivative

The derivative of $f(x)$ at a, written $f'(a)$, is the instantaneous rate of change of $f(x)$ when $x = a$.

Suppose $P(n)$ gives the total profit, in dollars, earned by selling n widgets. How do you interpret $P'(100)$?

Interpreting the Derivative

The derivative of $f(x)$ at a, written $f'(a)$, is the instantaneous rate of change of $f(x)$ when $x = a$.

Suppose $h(t)$ gives the height of a rocket t seconds after liftoff. What is the interpretation of $h'(t)$?

Interpreting the Derivative

The derivative of $f(x)$ at a, written $f'(a)$, is the instantaneous rate of change of $f(x)$ when $x = a$.

Suppose $M(t)$ is the number of molecules of a chemical in a test tube t seconds after a reaction starts. Interpret $M'(t)$.

Interpreting the Derivative

The derivative of $f(x)$ at a, written $f'(a)$, is the instantaneous rate of change of $f(x)$ when $x = a$.

Suppose $G(w)$ gives the diameter in millimetres of steel wire needed to safely support a load of w kg. Suppose further that $G'(100) = 0.01$. How do you interpret $G'(100) = 0.01$?

A paper[1] on the impacts of various factors in average life expectancy contains the following:

> *The only statistically significant variable in the model is physician density. The coefficient for this variable 20.67 indicating that a one unit increase in physician density leads to a 20.67 unit increase in life expectancy. This variable is also statistically significant at the 1% level demonstrating that this variable is very strongly and positively correlated with quality of healthcare received. This denotes that access to healthcare is very impactful in terms of increasing the quality of health in the country.*

[1] Natasha Deshpande, Anoosha Kumar, Rohini Ramaswami, *The Effect of National Healthcare Expenditure on Life Expectancy*, page 12.
Remark: physician density is measured as number of doctors per 1000 members of the population.

If $L(p)$ is the average life expectancy in an area with a density p of physicians, write the statement as a derivative: "a one unit increase in physician density leads to a 20.67 unit increase in life expectancy."

Equation of the Tangent Line

The tangent line to $f(x)$ at a has slope $f'(a)$ and passes through the point $(a, f(a))$.

Tangent Line Equation – Theorem 2.3.2

The tangent line to the function $f(x)$ at point a is:

$$(y - f(a)) = f'(a)(x - a)$$

Point-Slope Formula

In general, a line with slope m passing through point (x_1, y_1) has the equation:

$$(y - y_1) = m(x - x_1)$$

Find the equation of the tangent line to the curve $f(x) = \sqrt{x}$ at $x = 9$. (Recall $\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{x}\right] = \frac{1}{2\sqrt{x}}$).

Memorize

The tangent line to the function $f(x)$ at point a is:

$$(y - f(a)) = f'(a)(x - a)$$

Now You Let $s(t) = 3 - 0.8t^2$. Then $s'(t) = -1.6t$. Find the equation for the tangent line to the function $s(t)$ when $t = 1$.

Included Work

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Natasha Deshpande, Anoosha Kumar, Rohini Ramaswami. (2014). The Effect of National Healthcare Expenditure on Life Expectancy, page 12. *College of Liberal Arts - Ivan Allen College (IAC), School of Economics: Econometric Analysis Undergraduate Research Papers*. https://smartech.gatech.edu/handle/1853/51648 (accessed July 2021), 72