The University of British Columbia
Department of Mathematics
Qualifying Examination—Differential Equations

January 2023

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1. (10 points) You are attempting to solve for x_1, x_2, x_3 in the matrix equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -2 \\ 1 & 2 & -2 \\ 1 & 2 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

   First, find the projection $\hat{\mathbf{b}}$ on the column space of A. Then, solve the new system $A\mathbf{x} = \hat{\mathbf{b}}$ for x_1, x_2, x_3.

2. (10 points) Let A be an $n \times n$ diagonalizable matrix. Show that A and A^T are similar, i.e., there exists an invertible matrix M such that $A^T = MAM^{-1}$.

3. (10 points) Let A be a real $n \times n$ matrix such that $\|A\mathbf{v}\| = \|\mathbf{v}\|$ for all $\mathbf{v} \in \mathbb{R}^n$. Show that A is orthogonal.

4. (10 points) IVP-ODE problems
   (a) *[5 points]* Find the explicit solution $y(x)$ to the following IVP:

$$y' - \tan(x)y = 3x + 1\,, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\,, \quad y(0) = 1\,.$$

   (b) *[5 points]* Find the explicit solution $y(t)$ as a function of $k \in \mathbb{R}$ to the following IVP:

$$y'' - y' - 2y = 120e^{-t}\,, \quad y(0) = 215\,, \quad y'(0) = k.$$

   Then, determine the value of k for which $\lim_{t\to\infty} y(t) = 0$.

5. (10 points) We consider the Laplace equation on a circular wedge with the following boundary conditions:

$$\begin{aligned} \Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad &, \quad 0 \leq r \leq a \quad , \quad 0 \leq \theta \leq \pi/4 \\ u(r,0) = 1 \quad &, \quad u_\theta(r,\pi/4) = 0 \\ u(a,\theta) = \sin(2\theta) + 1 \quad &, \quad u \text{ remains finite as } r \to 0 \end{aligned}$$

   Here $a > 0$ is the radius of the wedge.

   (a) *[3 points]* We seek a solution of the form $u(r,\theta) = A + v(r,\theta)$ where $A \in \mathbb{R}$. Determine A and the problem solved by the function $v(r,\theta)$.

   (b) *[7 points]* Find $v(r,\theta)$ using the method of separation of variables and give the complete solution $u(r,\theta)$.

   Tip: the eigenvalues and eigenfunctions of the 1D negative Laplacian operator $-\Delta$ over $x \in [0, L]$ with homogeneous Dirichlet boundary condition at $x = 0$ and homogeneous Neumann boundary condition at $x = L$ are $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$ and $\sin\left(\frac{(2n-1)\pi}{2L}x\right)$, respectively, for $n = 1, 2, \ldots$

6. (10 points) Convolution and Laplace transform
   We define the convolution of two functions $f(t)$ and $g(t)$ as

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau\,.$$

   (a) *[5 points]* Prove the convolution theorem for Laplace tranforms

$$\mathcal{L}\{(f * g)(t)\}(s) = F(s)G(s)\,,$$

   where $F(s) = \mathcal{L}\{f(t)\}(s)$ and $G(s) = \mathcal{L}\{g(t)\}(s)$.

   (b) *[5 points]* Use the convolution theorem (and not any other method) to calculate the inverse Laplace transform of $H(s) = \dfrac{a}{s^2(s^2 + a^2)}$ for any $a \in \mathbb{R}\,,\ a \neq 0$.

Help with Laplace transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1. $\sin(xt)$	$\dfrac{x}{s^2 + x^2}, \quad s > 0$
2. $\cos(xt)$	$\dfrac{s}{s^2 + x^2}, \quad s > 0$
3. t^n, n positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$