1. (8 points) (a) (2 points) Find a basis for the space $M_2$ of $2 \times 2$ matrices.
   (b) (2 points) What is the dimension of $M_2$?
   (c) (2 points) Find a basis for the subspace $D_2$ of diagonal $2 \times 2$ matrices.
   (d) (2 points) What is the dimension of $D_2$?

2. (12 points) Let $x_1 = (-1, 1)$ and $x_2 = (1, 0)$ and consider the sets $S_1, S_2 \subset \mathbb{R}^2$ given by
   \[ S_1 = \{v : v^T x_1 = 0\} \]
   and
   \[ S_2 = \{v : v^T x_2 = 0\}. \]
   (a) (4 points) Find an expression for the projection operators $P_1$ onto $S_1$ and $P_2$ onto $S_2$.
   (b) (4 points) Let $y = (1, 2)$. Compute $P_2 y$, the projection of $y$ onto $S_2$, and $P_1 P_2 y$, and $P_2 P_1 P_2 y$.
   (c) (4 points) Find $\lim_{n \to \infty} (P_1 P_2)^n y$. Justify your answer.

3. (10 points) Let $M$ be a square matrix of size $n \times n$ with distinct eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_n$. Show that the determinant of $M$ is the product of its eigenvalues:
   \[ \det(M) = \prod_{i=1}^{n} \lambda_i. \]
4. (10 points) (a) [7 points] Let $n$ be a non-negative integer. **Legendre’s differential equation** is

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0, \quad -1 \leq x \leq 1,$$

and has the solution, $y(x) = P_n(x)$, which is regular at $x = \pm 1$ and satisfies

$$P_n(1) = 1 \quad \& \quad \int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{2n+1}.$$

Look for a solution to the differential equation with the form of a power series,

$$y = \sum_{j=0}^{\infty} c_j x^j.$$

Establish the recurrence relation satisfied by the coefficients $c_j$, and thereby argue that the solutions $P_n(x)$ take form of polynomials of finite degree if $n = 0, 1, 2, \ldots$ Hence find $P_n(x)$ for $n = 0, 1, 2, 3$.

(b) [3 points] Place Legendre’s equation in Sturm-Liouville form, and then show that

$$\int_0^1 P_n(x) dx = \frac{P_n'(0)}{n(n+1)}$$

for $n > 0$. Hence use Sturm-Liouville theory to establish that

$$\text{sgn}(x) = \sum_{n \text{ odd}} \frac{(2n + 1)P_n'(0)}{n(n+1)} P_n(x),$$

where $\text{sgn}(x) = 1$ if $x > 0$, $\text{sgn}(x) = -1$ if $x < 0$, and $\text{sgn}(0) = 0$.

5. (10 points) By either observing that $y(x) = x$ is a homogeneous solution to the differential equation or otherwise, solve the differential equation

$$x^2 y'' - \alpha (xy' - y) = x(1 + x), \quad y(0) = y(1) = 0,$$

where $\alpha$ is a positive parameter. Is the solution unique?

6. (10 points) Use separation of variables to solve Laplace’s equation

$$\frac{1}{r} (ru_r)_r + \frac{1}{r^2} u_{\theta \theta} = 0,$$

outside the unit disk, $r \geq 1$, subject to

$$u(1, \theta) = \begin{cases} 
0 & \theta = 0 \text{ or } \pi \\
\pi/2 & 0 < \theta < \pi \\
-\pi/2 & \pi < \theta < 2\pi
\end{cases}.$$
Using
\[ \frac{1}{2} \ln \left( \frac{1+\psi}{1-\psi} \right) = \sum_{n>0, n \text{ odd}} \frac{\psi^n}{n}, \]
sum the series for \( u(r, \theta) \), and hence write down a compact logarithmic expression for the solution.