The University of British Columbia
Department of Mathematics
Qualifying Examination—Analysis

September 2022

Real analysis

1. (10 points) Let S be the part of the graph

$$z = f(x,y) = x(5 - x^2 - y^2)^{317}$$

that is inside the cylinder $Z : x^2 + y^2 = 4$, with upward normal. Find the flux integral $J = \iint_S \operatorname{curl} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ for the vector field

$$\vec{F}(x,y,z) = \left((z-x)^{10}y + e^{x+z},\, x^2 z,\, (z-x)^{10}y + e^{x+z}\right).$$

2. (10 points) Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Recall a sequence $a = (a_k)_{k\in\mathbb{N}_0}$ is in ℓ^q, $0 < q < \infty$, if $\|a\|_{\ell^q} = (\sum_{k\in\mathbb{N}_0} |a_k|^q)^{1/q}$ is finite, $\|a\|_{\ell^\infty} = \sup_{k\in\mathbb{N}_0} |a_k|$, and the convolution of two sequences (a_k) and (b_k) is defined by $(a * b)_k = \sum_{j=0}^{k} a_j b_{k-j}$ for $k \in \mathbb{N}_0$.

   (a) Show that there is a sequence $\delta \in \ell^1$ such that $\delta * b = b$ for any $b \in \ell^q$, $0 < q \le \infty$. Note: For functions in $L^q(\mathbb{R}^n)$, such a δ is only a distribution, not a function.

   (b) Show that Young's convolution inequality

$$\|a * b\|_{\ell^q} \le \|a\|_{\ell^1} \|b\|_{\ell^q}$$

   fails if $q < 1$. You may assume the existence of δ in Part (a).

3. (10 points) Let $K(x,y)$ be a continuous function for $x, y \in [0,1]$.

   (a) If $f : [0,1] \to \mathbb{R}$ is continuous, prove that $F(x) = \int_0^1 K(x,y) f(y) dy$ defines a continuous function F on $[0,1]$.

   (b) Suppose $\alpha = \max_x \int_0^1 |K(x,y)| dy < 1$. Prove that there is a unique continuous function $f : [0,1] \to \mathbb{R}$ such that

$$f(x) = \sin x + \int_0^1 K(x,y) f(y) dy \quad \text{for all } x \in [0,1].$$

Complex analysis

4. (10 points) By using the Cauchy Residue Theorem, calculate the following integral. Justify your answer.

$$\int_0^\infty \frac{\log x}{x^2+2x+2}dx.$$

5. (10 points) (a) (4 points) Show that if $f(z)$ is analytic in an open and connected domain D and $|f(z)| = Constant$ in D, then $f(z)$ is constant.

   (b) (6 points) Count the number of zeros (where each zero is counted as many times as its multiplicity) of $f(z) = z^6 + 3z^3 + 4z^2 - 1$ in the annulus $1 \leq |z| \leq 2$.

6. (10 points) (a) (2 points) Find the image of the region $S = \{z : 0 \leq Re(z) \leq 1, -\frac{\pi}{2} \leq Im(z) \leq \pi\}$ under the map $f(z) = e^z$.

   (b) (8 points) Find a conformal map from the unbounded region outside the disks $\{|z+1| \leq 1\} \cup \{|z-1| \leq 1\}$ to the upper half plane $\{Im(z) > 0\}$.