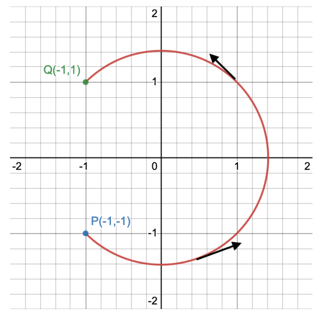
The University of British Columbia
Department of Mathematics
Qualifying Examination—Analysis
January 2023

Real analysis



1. (10 points) Let C be the right arc on the circle $x^2 + y^2 = 2$ from $P(-1, -1)$ to $Q(-1, 1)$ as shown in the picture. It is counterclockwise.

   Hint. $\int \sin^2 t\, dt = \frac{t}{2} - \frac{1}{4}\sin(2t)$, $\int \cos^2 t\, dt = \frac{t}{2} + \frac{1}{4}\sin(2t)$

   (a) (6 points) Evaluate $I_1 = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ for $\vec{\mathbf{F}} = (y, 3x)$.

   (b) (4 points) Find a potential for the conservative vector field $\vec{\mathbf{G}} = (x^3, y^2)$ and evaluate $I_2 = \int_C \vec{\mathbf{G}} \cdot d\vec{\mathbf{r}}$ along the same curve C.

2. (12 points) Are the following true or false? If true, give a proof; if false, provide a counterexample.

   (a) If $f, g : (0, 1) \to \mathbb{R}$ are uniformly continuous on $(0, 1)$, then so is $h(x) = f(x)g(x)$.

   (b) If $f, g : (0, 1) \to (0, \infty)$ are positive and uniformly continuous on $(0, 1)$, then so is $h(x) = f(x)/g(x)$.

   (c) If f is a continuously differentiable function on $[0, 1]$, then there is a sequence of polynomials $\{P_n : n \in \mathbb{N}\}$ such that P_n converges uniformly to f, and P_n' converges uniformly to f'.

3. (8 points) Prove the following inequality by the mean value theorem: Let $m > 0$. For some $C = C(m) > 0$,

$$(a + b)^m - a^m \leq C(b^m + ba^{m-1}), \quad \forall a, b > 0. \tag{1}$$

   Hint. Your proof should be valid for both $0 < m < 1$ and $1 \leq m < \infty$.

Complex analysis

4. (12 points) (a) (4 points) Find all solutions in the complex plane to the following equation:

$$\sin z = i\,,$$

where $z = x + iy$.

(b) (4 points) Find a branch cut for $\sqrt{z(z-1)}$ that is analytic in $\mathbb{C}\backslash[0,1]$ and takes the value $-\sqrt{2}$ at $z = 2$.

(c) (4 points) Calculate the following integral, providing justification for your result:

$$\int_{|z|=2} \frac{z^3}{z^5 + 3z + 1}\,.$$

(The contour is oriented counter-clockwise.)

5. (8 points) (a) (8 points) Use argument principle and Nyquist criterion to show that there are no zeroes of $p(z) = z^3 + z^2 + 4z + 1$ in $\{\mathrm{Re}(z) \geq 0\}$.

6. (10 points) (a) (4 points) Let f be analytic in $D = \{|z| \leq 2\}$. Assume that $|f(z)| \leq 1$ for $|z| = 2$. Show that

$$|f''(1)| \leq 2\,.$$

(b) (6 points) Let $f_n(z)$ be a sequence of analytic functions in an open connected domain D. Assume that $f_n(z)$ converges uniformly on a compact set of D to $f(z)$. Show that the sequence of derivatives $f_n'(z)$ also converges uniformly on a compact set of D to $f'(z)$.