The University of British Columbia
Department of Mathematics
Qualifying Examination—Analysis
January 2022

Real analysis

1. (10 points) Evaluate

$$\oint_C -x^2y\,dx \,+\, (e^{y^2} + xy^2)\,dy,$$

where C is the boundary of the half disk $0 \le y \le \sqrt{1-x^2}$, oriented counterclockwise.

2. (10 points) Determine if the following assertions are **true** or **false**, justifying the answers carefully.

Let

$$A = \{(x,y) \in \mathbb{R}^2 : y = qx \text{ for some } q \in \mathbb{Q}\}.$$

(a) (3 points) The set A is bounded.

(b) (3 points) The set A is closed.

(c) (4 points) The set A is connected.

3. (10 points) *Note*: In this problem, if you do the second part, you do not need to write up the first part separately, since it is a special case of the second part. The first part is meant to give you an additional chance for partial credit and to help you think about the second part.

(a) (4 points) Prove that if $f : [0,1] \to \mathbb{R}$ is a continuous function, then there is a sequence of polynomials $(Q_n)_{n=1}^{\infty}$ such that Q_n converges uniformly to f on $[0,1]$, and $Q_n(0) = f(0)$ for every $n \ge 1$.

(b) Prove that if $f : [0,1] \to \mathbb{R}$ is a continuous function, then there is a sequence of polynomials $(Q_n)_{n=1}^{\infty}$ such that Q_n converges uniformly to f on $[0,1]$, $Q_n(0) = f(0)$ for every $n \ge 1$, and $Q_n(1) = f(1)$ for every $n \ge 1$.

Complex analysis

4. (10 points) Compute the contour integral

$$\oint_C \frac{1}{z^2 \sin(z)}\, dz,$$

where C denotes the unit circle $\{z : |z| = 1\}$ traversed once in the counterclockwise direction.

5. (10 points) (a) (3 points) Show that the function $u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$ is **harmonic** in $D = \mathbb{R}^2 \setminus \{(0, 0)\}$. Here ln denotes the **real valued** natural logarithm.

   (b) (7 points) Does the function $u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$ have a **harmonic conjugate** in $D = \mathbb{R}^2 \setminus \{(0, 0)\}$? If yes, find a harmonic conjugate of u and if no, explain why u does not have a harmonic conjugate.

6. (10 points) (a) (3 points) Find the Laurent series of the function $f(z) = \frac{1}{(z-1)(z-2)}$ in the annulus $\{z : 1 < |z| < 2\}$.
   Note: Your answer should contain all terms and not just the first few terms. Express the answer as $\Sigma_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} z^{-n}$.

   (b) (4 points) Show that there is no analytic function $f : D \to \mathbb{C}$ such that $f^{(j)}(0) = j!$ for all $j \in \mathbb{N}$, where $f^{(j)}$ denotes the j-th derivative of f and $D = \{z : |z| < 3\}$.

   (c) (3 points) If f is analytic in the annulus $\{z : 1 \leq |z| \leq 2\}$ such that $|f(z)| \leq 3$ on $|z| = 1$ and $|f(z)| \leq 12$ on $|z| = 2$. Then show that $|f(z)| \leq 3\,|z|^2$ for all z such that $1 \leq |z| \leq 2$.