The University of British Columbia
Department of Mathematics
Qualifying Examination—Analysis
September, 2023

Real analysis

1. (10 points) Let S be the part of the paraboloid $z = 2 - x^2 - y^2$ above the cone $z = \sqrt{x^2 + y^2}$, with upward orientation. Let
$$\mathbf{F} = (\tan\sqrt{z} + \sin(y^3))\,\mathbf{i} + e^{-x^2}\,\mathbf{j} + z\,\mathbf{k}.$$
Evaluate the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

2. (10 points) Let $f(x) = \sum_{n=1}^{\infty} \sin(nx)x^n$ for those x for which the series converges. Note that this is NOT a power series.
   (a) (5 points) Show that f is defined and continuous on $(-1, 1)$.
   (b) (5 points) Show that f is differentiable and that f' is continuous on $(-1, 1)$

3. (10 points) Let $\{x_n\}$ be a sequence of positive real numbers, and define
$$\alpha = \liminf_{n\to\infty} \frac{x_{n+1}}{x_n}, \qquad \beta = \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$
(Note that $\alpha = \infty$ and/or $\beta = \infty$ may occur.)
   (a) (2 points) Prove that if $\beta < 1$, the sequence $\{x_n\}$ converges.
   (b) (6 points) Prove that if $\alpha > 1$, the sequence $\{x_n\}$ diverges.
   (c) (1 point) Give an example of a *convergent* sequence $\{x_n\}$ for which $\alpha = 1/2$.
   (d) (1 point) Give an example of a *divergent* sequence $\{x_n\}$ for which $\beta = 1$.

Complex analysis

4. (10 points) (a) Find
$$\int_C \left(\frac{z}{(z-1)(z^2+1)} + \frac{e^z}{z - 3i} \right) dz$$
where C is the counterclockwise oriented circle centred at $(0, 0)$ of radius 2.
   (b) Find all values of z in $\mathbb{C}$ such that $f(z) = 2(x^3 - 3xy^2 + y) + i(3yx^2 - y^3)$ is analytic at z.

5. (10 points) (a) Find the domain of analyticity of $f(z) = \sqrt{\operatorname{Log}(z+1) - \frac{\pi}{2}i}$, where the square root is given by the principal branch and $\operatorname{Log} z$ is the principal branch of $\log z$. (b) Find $f(-i)$.

6. (10 points) Suppose that f is an analytic function on $H = \{z \in \mathbb{C} : Re(z) \le 0\}$ with
$$f(-1) = f'(-1) = 0 \text{ and } f''(-1) = \frac{i}{2}.$$
   (a) Show that $g(z) = f(z)/(z+1)^2$ is analytic on H. Find the residue of g at -1 and the residue of $f/(z+1)^3$ at -1.
   (b) Suppose $|f(z)| \le \frac{1}{2}|z+1|^2$. Show $\left|f(-\frac{3}{2})\right| \le \frac{9}{80}$.