The University of British Columbia
Department of Mathematics
Qualifying Examination—Algebra
September 2022

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1. (8 points) Find the shortest distance from x to $U = \text{span}\{u_1, u_2\} \subseteq \mathbb{R}^4$ where

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

2. (8 points) Let A be a real 3×3 matrix and suppose that the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are eigenvectors of A. Show that A is symmetric.

3. (14 points) Recall the matrix norm $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$.

   (a) (7 points) Let A be an $n \times n$ real matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, and singular values $\sigma_1, \ldots, \sigma_n$. What is $\|A\|$? Justify your answer.

   (b) (7 points) Determine the matrix norm $\|A\|$ for the matrix

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}.$$

In this section, you can use any theorem or fact from group theory, commutative algebra, Galois theory, etc. without proof as long as you state it clearly.

4. (8 points) Let N be the submodule of $\mathbb{Z}^3$ generated by the vectors $\langle 2, 2, 4 \rangle$, $\langle 4, -6, 12 \rangle$, and $\langle 2, -8, 8 \rangle$.
   (a) (2 points) Give the definition of a *free* module over $\mathbb{Z}$.
   (b) Is N a free module? If yes, find the rank of N.
   (c) Find the quotient $\mathbb{Z}^3/N$.

5. (10 points) Assume R is a commutative ring with 1. Recall that an element $x \in R$ is called *nilpotent* if $x^n = 0$ for some $n \in \mathbb{Z}^+$.
   (a) Prove that the set of nilpotent elements forms an ideal, called the *nilradical* $\mathcal{N}(R)$.
   (b) Prove that the following are equivalent:
      (i) R has exactly one prime ideal
      (ii) every element of R is either nilpotent or a unit
      (iii) $R/\mathcal{N}(R)$ is a field

6. (12 points) (a) List all the groups of order 6 up to isomorphism (with proof that the list is exhaustive; you can use the theorems of group theory without proof).
   (b) Find the Galois group of the splitting field of the polynomial $f(X) = X^3 - 7$ over $\mathbb{Q}$.
   (c) Is $\mathbb{Q}(7^{1/3})$ Galois over $\mathbb{Q}$?