The University of British Columbia
Department of Mathematics
Qualifying Examination—Algebra
January 2022

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1. (8 points) (a) (2 points) Find a basis for the space M_2 of 2×2 matrices.
   (b) (2 points) What is the dimension of M_2?
   (c) (2 points) Find a basis for the subspace D_2 of diagonal 2×2 matrices.
   (d) (2 points) What is the dimension of D_2?

2. (12 points) Let $x_1 = (-1, 1)$ and $x_2 = (1, 0)$ and consider the sets $S_1, S_2 \subset \mathbb{R}^2$ given by

$$S_1 = \{v : v^T x_1 = 0\}$$

   and

$$S_2 = \{v : v^T x_2 = 0\}.$$

   (a) (4 points) Find an expression for the projection operators P_1 onto S_1 and P_2 onto S_2.
   (b) (4 points) Let $y = (1, 2)$. Compute $P_2 y$, the projection of y onto S_2, and $P_1 P_2 y$, and $P_2 P_1 P_2 y$.
   (c) (4 points) Find $\lim_{n\to\infty} (P_1 P_2)^n y$. Justify your answer.

3. (10 points) Let M be a square matrix of size $n \times n$ with distinct eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_n$. Show that the determinant of M is the product of its eigenvalues: $\det(M) = \prod_{i=1}^{n} \lambda_i$.

4. (8 points) Let Γ be a cyclic group of order 2^n for some integer $n > 0$.

   (a) Prove that when $n = 1$ the group Γ is unique up to isomorphism.

   (b) Prove that the group G of isomorphisms $\Gamma \to \Gamma$ is a group of order 2^{n-1}.

   (c) When $n = 3$, prove that G from part (b) is not a cyclic group.

5. (10 points) Let R be a principal ideal domain and let $I \subset R$ be a proper ideal.

   (a) If I is prime, show that R/I is a principal ideal domain.

   (b) If I is not prime, show that R/I is not a unique factorization domain.

   (c) If $\mathbf{k}$ is a field, prove that $\mathbf{k}[x]$ is a principal ideal domain. You should assume the existence of the division algorithm for this part.

   (d) Show that no quotient of $\mathbb{Q}[x]$ is isomorphic to $\mathbb{Z}[x]$.

6. (12 points) For this problem, $\mathbf{k}$ is a field, with $\mathbf{k}^\times = \mathbf{k} \setminus \{0\}$ the (multiplicative) group of units, and $\mathbb{Z}$ denotes the integers. Recall that elements of $GL(2, \mathbf{k})$ are 2×2 invertible matrices with coefficients in $\mathbf{k}$. You will need the following definition: A *valuation* is a surjective group homomorphism $v \colon \mathbf{k}^\times \to \mathbb{Z}$ satisfying

$$v(a + b) \geq \min\{v(a), v(b)\}$$

   whenever $a, b, a + b \in \mathbf{k}^\times$.

   (a) Show that if $v(a) \neq v(b)$ then $v(a + b) = \min\{v(a), v(b)\}$.

   (b) Show that the set $\mathcal{O}_v = \{a \in \mathbf{k} \mid v(a) \geq 0\}$ is a subring of $\mathbf{k}$ and determine its group of units.

   (c) Let $V = \mathbf{k}^2$ be the standard 2-dimensional vector space. Observe that $\mathcal{O}_v$ acts by multiplication on V in such a way that for $a, b \in \mathcal{O}_v$ and for $x, y \in V$ we have

      - $a(x + y) = ax + ay$;
      - $(a + b)x = ax + ab$; and
      - $(ab)x = a(bx)$.

      This certifies that V is a *module* over $\mathcal{O}_v$. A *lattice* in this setting is a $\Lambda \subseteq V$ such that for any $x \in \Lambda$ and any $a \in \mathcal{O}_v$ the product ax is in Λ. A basic fact, that you may assume, is that every Λ is free as a module over $\mathcal{O}_v$ with free basis $\{e, f\}$, in the sense that every $x \in \Lambda$ may be uniquely expressed as $x = \alpha e + \beta f$ for $\alpha, \beta \in \mathcal{O}_v$, such that $\text{span}_{\mathbf{k}}\{e, f\} = V$ as a $\mathbf{k}$ vector space. Verify that for any two lattices Λ_0 and Λ_1 there is an element of $GL(2, \mathbf{k})$ carrying Λ_0 bijectively to Λ_1.

   (d) Show that any lattice automorphism is given by an element in $GL(2, \mathcal{O}_v)$.

   (e) Given lattices Λ_0 and Λ_1 define $\delta(\Lambda_0, \Lambda_1) = v(\det(A))$ where A carries Λ_0 bijectively to Λ_1. Prove that the quantity $\delta(\Lambda_0, \Lambda_1)$ does not depend on the choice of matrix A.

   (f) Verify that $\delta(\Lambda_0, \Lambda_2) = \delta(\Lambda_0, \Lambda_1) + \delta(\Lambda_1, \Lambda_2)$ and that $\delta(\Lambda, \Lambda) = 0$.