PUTNAM PRACTICE SET 7

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Problem 1. Let $P \in \mathbb{C}[x]$ be a polynomial of degree $n \geq 1$ with the property that $P(k) = \frac{1}{\binom{n+1}{k}}$ for each $k = 0, 1, \dots, n$. Find $P(n+1)$.

Solution. Using the Lagrange inetrpolation:

$$\begin{aligned} &P(x) \\ &= \prod_{i=0}^{n}(x-i) \cdot \sum_{k=0}^{n} \frac{P(k) \cdot \prod_{j \neq k}(k-j)^{-1}}{x-k} \\ &= \prod_{i=0}^{n}(x-i) \cdot \sum_{k=0}^{n} \frac{P(k)}{(-1)^{n-k}(k-1)! \cdot (n-k)! \cdot (x-k)} \end{aligned}$$

and so,

$$\begin{aligned} &P(n+1) \\ &= (n+1)! \cdot \sum_{k=0}^{n} \frac{k! \cdot (n+1-k)!}{(n+1)! \cdot (-1)^{n-k} \cdot k! \cdot (n-k)! \cdot (n+1-k)} \\ &= \sum_{k=0}^{n}(-1)^{n-k} \\ &= \frac{1+(-1)^n}{2}. \end{aligned}$$

Problem 2. Find the maximum value for $m^2 + n^2$ where $1 \leq m, n \leq 2019$ and moreover $(n^2 - mn - m^2)^2 = 1$.

Solution. First we observe that if $1 \leq n \leq m$, then

$$|n^2 - nm - m^2| \geq m^2 \geq 1,$$

with equality only if $(m, n) = (1, 1)$. So, $n > m$ unless $(m, n) = (1, 1)$. Next, we observe that if we dneote by S the set of all pairs (m, n) satisfying the given relation, then $(m, n) \in S \setminus \{(1, 1)\}$ yields that $(n - m, m) \in S$ because

$$m^2 - (n-m)m - (n-m)^2 = m^2 + nm - n^2.$$

Hence, starting with any element in S, after finitely many steps we arrive at the pair $(1, 1)$. Conversely, in order to generate any pair $(m, n) \in S$ we can start from $(1, 1)$ and then apply finitely many times the transformation

$$(a, b) \mapsto (b, a + b).$$

So, this means that for any positive integer ℓ, we have that $(F_\ell, F_{\ell+1})$ are all the elements in S, where $\{F_\ell\}$ is the Fibonacci sequence. So, the largest value for

m^2+n^2 is obtained by determining the largest ℓ such that $F_{\ell+1}\le 2019$. We have that

$$F_0=0;\ F_1=F_2=1;\ F_3=2;\ F_4=3;\ F_5=5$$
$$F_6=8;\ F_7=13;\ F_8=21;\ F_9=34$$
$$F_{10}=55;\ F_{11}=89;\ F_{12}=144;\ F_{13}=233$$
$$F_{14}=377;\ F_{15}=610;\ F_{16}=987$$
$$F_{17}=1597 \text{ and } F_{18}=2584.$$

Hence the largest value is obtained for $\ell=16$ and so, the largest value for m^2+n^2 is 987^2+1597^2.

Problem 3. We define the recurrence sequence $\{a_n\}_{n\ge 1}$ given by:

$$a_1=1 \text{ and } a_{n+1}=\frac{1+4a_n+\sqrt{1+24a_n}}{16} \text{ for each } n\ge 1.$$

Find a_{2019}.

Solution. We compute

$$a_2=\frac{1+4+\sqrt{25}}{16}=\frac{5}{8}$$

and then

$$a_3=\frac{1+\frac{5}{2}+\sqrt{16}}{16}=\frac{15}{32}$$

and also

$$a_4=\frac{1+\frac{15}{8}+\sqrt{\frac{49}{4}}}{16}=\frac{51}{128}$$

and just to make sure the pattern holds:

$$a_4=\frac{1+\frac{51}{32}+\sqrt{\frac{169}{16}}}{16}=\frac{119}{64}.$$

So, we let $1+24a_n=b_n^2$ and thus

$$\frac{b_{n+1}^2-1}{24}=a_{n+1}=\frac{1+\frac{b_n^2-1}{6}+b_n}{16}=\frac{b_n^2+6b_n+5}{96}$$

and so,

$$4b_{n+1}^2-4=b_n^2+6b_n+5$$

and thus

$$(2b_{n+1})^2=(b_n+3)^2.$$

So, $b_{n+1}=\frac{b_n+3}{2}$, which (knowing that $b_1=5$) leads us to

$$b_{n+1}-3=\frac{b_n-3}{2}=\frac{b_1-3}{2^n}=\frac{1}{2^{n-1}}.$$

Therefore

$$a_n=\frac{b_n^2-1}{24}=\frac{8+\frac{3}{2^{n-3}}+\frac{1}{2^{2n-4}}}{24}=\frac{1}{3}+\frac{1}{2^n}+\frac{1}{3\cdot 2^{2n-1}}.$$

Problem 4. Let $1\le r\le n$ be integers. We consider the set $\mathcal{M}$ the set of all subsets of $\{1,2,\ldots,n\}$ consisting of exactly r elements. For each $S\in\mathcal{M}$, we let

m_S be the smallest element contained in S. Find the arithmetic mean of all m_S (for $S \in \mathcal{M}$).

Solution. There are $\binom{n}{r}$ elements in $\mathcal{M}$. Now, an element is the smallest in a set with r elements has to be at most $n-r+1$. Now, for each $i \in \{1, \ldots, n-r+1\}$, there are precisely $\binom{n-i}{r-1}$ sets for which i is the smallest element in that set. So, the arithmetic mean of all m_S is

$$\frac{\sum_{i=1}^{n-r+1} i \cdot \binom{n-i}{r-1}}{\binom{n}{r}}$$

$$= \frac{(n+1) \cdot \sum_{i=1}^{n-r+1} \binom{n-i}{r-1} - \sum_{i=1}^{n-r+1} (n-i+1) \cdot \binom{n-i}{r-1}}{\frac{n!}{r!(n-r)!}}$$

$$= \frac{(n+1) \cdot \binom{n}{r} - r \cdot \sum_{i=1}^{n-r+1} \binom{n-i+1}{r}}{\frac{n!}{r!(n-r)!}}$$

$$= \frac{\frac{(n+1)!}{r!(n-r)!} - \frac{r \cdot (n+1)!}{(r+1)!(n-r)!}}{\frac{n!}{r!(n-r)!}}$$

$$= (n+1) - \frac{r(n+1)}{r+1}$$

$$= \frac{n+1}{r+1}.$$