PUTNAM PRACTICE SET 6

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Problem 1. Let a and s be real numbers satisfying the following properties:

- $0 < a \leq 1$; and
- $s > 0$, but $s \neq 1$.

Prove that $\frac{1-s^a}{1-s} \leq (1+s)^{a-1}$.

Solution. We observe that replacing s by $1/s$ yields the same inequality since

$$\frac{1-\frac{1}{s^a}}{1-\frac{1}{s}} = \frac{1}{s^{a-1}} \cdot \frac{1-s^a}{1-s}$$

and $(1+1/s)^{a-1} = \frac{1}{s^{a-1}} \cdot (1+s)^{a-1}$. So, from now on, we assume $0 < s < 1$. Also, we may assume $0 < a < 1$ because the case $a = 1$ is clear. We let

$$f_a(s) := -(1-s^a) + (1-s) \cdot (1+s)^{a-1} \text{ for } 0 < s < 1.$$

We observe that $f_a(0) = 0 = f_a(1)$ and so, in order to prove that $f_a(s) > 0$ for $0 < s < 1$, it suffices to prove that there exists a unique $d \in (0,1)$ such that f_a is increasing on $(0,d)$ and then f_a is decreasing on $(d,1)$; this will guarantee that $f_a(s) > 0$ for all $0 < s < 1$.

So, we compute

$$\begin{aligned}
&f_a(s)' \\
&\quad = as^{a-1} - (1+s)^{a-1} + (1-s) \cdot (a-1) \cdot (1+s)^{a-2} \\
&\quad = s^{a-1} \cdot \left(a - (1+1/s)^{a-1} - (1-1/s) \cdot (a-1) \cdot (1+1/s)^{a-2}\right) \\
&\quad = s^{a-1} \cdot g_a(1/s),
\end{aligned}$$

where $g_a(x) := a - (1+x)^{a-1} - (1-x) \cdot (a-1) \cdot (1+x)^{a-2}$, which is defined for $x > 1$ (note that x corresponds to $1/s$, where $0 < s < 1$). Once again we differentiate:

$$\begin{aligned}
&g_a'(x) \\
&\quad = -(a-1) \cdot (1+x)^{a-2} + (a-1) \cdot (1+x)^{a-2} - (1-x) \cdot (a-1) \cdot (a-2)(1+x)^{a-3} \\
&\quad = -(a-1)(a-2) \cdot (1-x) \cdot (1+x)^{a-3} > 0
\end{aligned}$$

since $0 < a < 1$ and $x > 1$. On the other hand, $g_a(1) = a - 2^{a-1}$ and we view it as a function of a, i.e.,

$$h(a) := a - 2^{a-1} \text{ for } 0 < a < 1.$$

We have

$$h'(a) = 1 - \ln(2) \cdot 2^{a-1},$$

which is decreasing and its smallest value is obtained for $a = 1$ and then

$$h'(a) > h'(1) = 1 - \ln(2) > 0 \text{ for } 0 < a < 1.$$

So, $h(x)$ is increasing and $h(a) < h(1) = 0$ for all $0 < a < 1$. In conclusion, $g_a(1) < 0$ and because g_a is increasing, in order to determine the sign of $g_a(x)$, we need to compute

$$\lim_{x\to\infty} g_a(x) = a > 0$$

because $0 < a < 1$. In conclusion, there exists some $c \in (1,\infty)$ such that $g_a(x) < 0$ for all $x \in (1,c)$ and $g_a(x) > 0$ for all $x \in (c,\infty)$. Hence, letting $d := 1/c \in (0,1)$, we obtain that $f_a'(s) > 0$ for all $s \in (0,d)$ and $f_a'(s) < 0$ for all $s \in (d,1)$. This concludes our proof that $f_a(s) > 0$ for all $0 < s < 1$ (since $f_a(s)$ increases on $(0,d)$ starting from $f(0) = 0$ and then it decreases on $(d,1)$ ending at $f_a(1) = 0$).

Problem 2. Let S be the set of all real numbers of the form $\frac{m+n}{\sqrt{m^2+n^2}}$ where m and n are positive integers. Prove that for each two distinct elements $u < v$ contained in S, there exists another element $w \in S$ such that $u < w < v$.

Solution. We observe that letting $r := \frac{m}{n}$ (where $m \le n$), then

$$\frac{m+n}{\sqrt{m^2+n^2}} = \frac{1+r}{\sqrt{1+r^2}}.$$

So, we let $f(r) := \frac{1+r}{\sqrt{1+r^2}}$ for all rational numbers $0 < r \le 1$. Now, we observe that the above function f is increasing since

$$f'(x) = \frac{1\cdot\sqrt{1+x^2} - (1+x)\cdot\frac{2x}{2\sqrt{1+x^2}}}{1+x^2} = \frac{1-x}{(1+x^2)^{\frac{3}{2}}} > 0$$

if $0 < x < 1$. So, for any distinct elements $u < v$ in S, there exist $0 < r_1 < r_2 \le 1$ such that $u = f(r_1)$ and $v = f(r_2)$. Hence, $w := f\left(\frac{r_1+r_2}{2}\right) \in S$ and $u < w < v$.

Problem 3. We consider a set S of finitely many disks in the cartesian plane (of arbitrary centers and arbitrary radii) and we let A be the area of the region represented by their union. Prove that there exists a subset $S_0 \subseteq S$ satisfying the following two properties:

- any two disks from S_0 are disjoint.
- the sum of the areas of the disks from S_0 is at least $\frac{A}{9}$.

Solution. We order the radii of the given disks $D_1, \dots, D_n$ in decreasing order $r_1 \ge r_2 \ge \cdots \ge r_n$. Also, we let D_i' (for $i = 1, \dots, n$) be the disks with the same centers O_i as the corresponding disk D_i but with its radii $r_i' := 3r_i$ for $i = 1, \dots, n$. Now, we select the disks D_{i_j} for $j = 1, \dots, m$ as follows: i_j is the smallest index k with the property that D_k is not contained in $\bigcup_{\ell<j} C_{i_\ell}'$. So, $i_1 := 1$ and clearly, $m \le n$ (i.e., the above process is destined to end in finitely many steps and at one moment there is no additional disk we can select in our process). Hence,

$$\bigcup_{i=1}^{n} D_i \subseteq \bigcup_{j=1}^{m} D_{i_j}'$$

and so, the area of $\bigcup_{j=1}^m D_{i_j}$ is $\frac{1}{9}$ times the area of $\bigcup_{j=1}^m D_{i_j}'$ and so, the area of $\bigcup_{i=1}^m D_{i_j}$ is at least $\frac{1}{9}$ times the area of $\bigcup_{i=1}^n D_i$.

On the other hand, we claim that there are no points in common for the disks $D_{i_1}, \dots, D_{i_m}$. Indeed, if there is a point x in common for the disks D_{i_k} and D_{i_ℓ} for $k < \ell$, then we have that for each point y in the disk D_{i_ℓ},

$$\begin{aligned}
&\operatorname{dist}(O_{i_k}, y) \\
&\qquad \le \operatorname{dist}(O_{i_k}, x) + \operatorname{dist}(x, O_{i_\ell}) + \operatorname{dist}(O_{i_\ell}, y) \\
&\qquad \le r_{i_k} + r_{i_\ell} + r_{i_\ell} \\
&\qquad \le r_{i_k} + 2r_{i_\ell} \\
&\qquad \le 3r_{i_k},
\end{aligned}$$

which yields that y is contained in D'_{i_k}. In other words, D_{i_ℓ} is contained in D'_{i_k}, where $k < \ell$; this contradicts our choice of i_ℓ which has the property that D_{i_ℓ} is not contained in $\bigcup_{j<\ell} D'_{i_j}$. In conclusion, the disks D_{i_j} (for $j = 1, \dots, m$) are indeed disjoint and the sum of their areas is at least $\frac{1}{9}$ times the area of the union of all disks $D_1, \dots, D_n$.

Problem 4. Let $\{u_n\}_{n\ge 1}$ be a recurrence sequence defined by $u_{n+1} = \frac{\sqrt[3]{64u_n+15}}{4}$ for each $n \ge 1$. Find $\lim_{n\to\infty} u_n$.

Solution. If there exists a limit L to the above sequence, then we must have

$$L = \frac{\sqrt[3]{64L+15}}{4},$$

i.e., $64L^3 = 64L + 15$, which suggests that the sequence either converges to one of the roots of the above equation, or that the sequence diverges to $\pm\infty$. On the other hand, since the function $f(x) := \frac{\sqrt[3]{64x+15}}{4}$ is increasing, we get that the relation between u_1 and u_2 determines whether the sequence is either increasing, or decreasing for all n, i.e., if $u_1 < u_2$, then $u_n < u_{n+1}$ (for all n), and if $u_1 > u_2$ then $u_n > u_{n+1}$ (for all n). Now, the roots of the equation

$$64x^3 - 64x - 15 = 0$$

are $x_2 = -\frac{1}{4}$ and $x_1 = \frac{1-\sqrt{61}}{8}$ and $x_3 = \frac{1+\sqrt{61}}{8}$. So, we have several cases:

Case 1. If $u_1 = x_i$ for some $i = 1, 2, 3$, then $x_n = u_i$ for all n and therefore, the limit is simply u_i in this case.

Case 2. If $u_1 < x_1$, then $u_2 = f(u_1) < f(x_1) = x_1$ but also $u_2 > u_1$, which means that the sequence $\{u_n\}$ converges to x_1 in this case.

Case 3. If $x_1 < u_1 < x_2$ then $x_1 < u_n < x_2$ for all n and moreover, $u_2 < u_1$ and so, $u_{n+1} < u_n$ for all n. Therefore, the sequence $\{u_n\}$ converges to x_1 in this case.

Case 4. If $x_2 < u_1 < x_3$ then $x_2 < u_n < x_3$ for all n and moreover, $u_1 < u_2$ and so, $u_n < u_{n+1}$ for all n. Therefore, the sequence $\{u_n\}$ converges to x_3.

Case 5. If $x_3 < u_1$ then $x_3 < u_2 < u_1$ and so, $u_{n+1} < u_n$ for all n. In conclusion, $\{u_n\}$ converges to x_3.