PUTNAM PRACTICE SET 2

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Problem 1. Consider the two sequences $\{a_m\}_{m\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ defined by

$$a_1 = 3 \text{ and for each } m \geq 1, \text{ we have } a_{m+1} = 3^{a_m}$$

and

$$b_1 = 100 \text{ and for each } n \geq 1, \text{ we have } b_{n+1} = 100^{b_n}.$$

Find the smallest possible integer n such that $b_n > a_{2019}$.

Solution. Clearly, $b_1 > a_2 = 27$ and then an easy induction yields that $b_n > a_{n+1}$ for all $n \geq 1$. Next we prove the following (surprising) result.

Claim 0.1. *For each $n \geq 1$, we have that $b_n < a_{n+2}$.*

Proof of Claim 0.1. Actually, we'll prove by induction an even stronger claim:

$$a_{n+2} > 2 + 5b_n \text{ for each } n \geq 1. \tag{1}$$

Inequality (1) holds (easily) for $n = 1$ and then, using the inductive hypothesis, we get

$$a_{n+3} = 3^{a_{n+2}} > 3^{2+5b_n} = 9 \cdot 243^{b_n} > 9 \cdot b_{n+1} > 2 + 5b_{n+1},$$

as claimed. This concludes our proof of Claim 0.1. □

Clearly, Claim 0.1 (coupled with the easy inequality $b_n > a_{n+1}$) yields that $b_{2017} < a_{2019} < b_{2018}$ and so, the desired integer in this problem is 2018.

Problem 2. Let $n > 1$ be an integer and let $a > 0$ be a real number. Let $x_1, \dots, x_n$ be nonnegative real numbers satisfying: $\sum_{i=1}^n x_i = a$. Find the maximum of $\sum_{i=1}^{n-1} x_i x_{i+1}$.

Solution. Let $x := \max_{i=1}^n x_i$. Then

$$\sum_{i=1}^{n-1} x_i x_{i+1} \leq x(a-x) \leq \frac{a^2}{4}$$

with equality if (for example) $x_1 = x_2 = \frac{a}{2}$.

Problem 3. Let N be the number of integer solutions to the equation $x^3 - y^3 = z^5 - t^5$ with the property that $0 \leq x, y, z, t \leq 2019^{2019}$. Let M be the number of integer solutions to the equation $x^3 - y^3 = z^5 - t^5 + 1$ with the property that $0 \leq x, y, z, t \leq 2019^{2019}$. Prove that $N > M$.

Solution. For each $0 \leq i \leq 2019^{3\cdot 2019} + 2019^{5\cdot 2019} := L$, we let n_i be the number of integers $0 \leq a, b \leq 2019^{2019}$ with the property that $a^3 + b^5 = i$. Then

$$N = n_0^2 + n_1^2 + \cdots + n_L^2$$

and $M = n_0n_1 + n_1n_2 + \cdots + n_{L-1}n_L$. Then we see that

$$N - M = \frac{n_0^2 + (n_0 - n_1)^2 + (n_1 - n_2)^2 \cdots + (n_{L-1} - n_L)^2 + n_L^2}{2} > 0$$

since $n_0 = n_L = 1$.

Problem 4. Find all $n \in \mathbb{N}$ such that $2^8 + 2^{11} + 2^n$ is a perfect square.

Solution. If $n \geq 8$, then letting $x := n - 8$ then we need that

$$\left(2^4\right)^2 \cdot (9 + 2^x)$$

be a perfect square, which is equivalent with $9 + 2^x$ be a perfect square y^2. Thus

$$2^x = (y - 3)(y + 3)$$

and so, both $y - 3$ and $y + 3$ are powers of 2 which yields that the only possibility is

$$y - 3 = 2^1 \text{ and } y + 3 = 2^3,$$

i.e., $y = 5$ and hence $x = 4$. So, $n = 12$; note that

$$2^8 + 2^{11} + 2^{12} = 80^2.$$

Now, if $n < 8$ then $2^8 + 2^{11} + 2^n$ is divisible by 2^n but not by 2^{n+1}; thus n must be even. So, we only need to check $n \in \{2, 4, 6\}$ and since

$$1 + 2^6 + 2^9 = 577 \text{ is not a perfect square}$$

$$1 + 2^4 + 2^7 = 145 \text{ is not a perfect square}$$

$$1 + 2^2 + 2^5 = 37,$$

we conclude that $n = 12$ is the only solution.