PUTNAM PRACTICE SET 11

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Problem 1. Find the sum of the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n}{3^m(3^mn+3^nm)}.$$

Solution. We let $a_n := \frac{n}{3^n}$ and then we notice that our series is precisely

$$S := \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^2a_n}{a_n+a_m}.$$

Clearly, since the series is absolutely convergent,

$$2S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^2a_n}{a_n+a_m}+\frac{a_ma_n^2}{a_m+a_n} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_ma_n = \left(\sum_{n=1}^{\infty}a_n\right)^2.$$

Now, the series $\sum_{n=1}^{\infty}\frac{n}{3^n}$ represents $f'(1)$ for the function

$$f(x) = \sum_{n=1}^{\infty}\frac{x^n}{3^n} = \frac{x}{3}\cdot\frac{1}{1-\frac{x}{3}} = \frac{x}{3-x}.$$

So, $f'(x) = \frac{3}{(3-x)^2}$ and therefore, $f'(1) = \frac{3}{4}$; so, we conclude that $S = \frac{9}{32}$.

Problem 2. Prove that there exists a positive ocnstant C such that for any polynomial $P \in \mathbb{R}[x]$ of degree less than 2020, we have that

$$P(0) \le C\cdot\int_{-1}^{1}|P(x)|\mathrm{dx}.$$

Solution. First, we note that if $P(0) = 0$, then any positive constant C would work. So, from now on, assume $P(0) \neq 0$, i.e., 0 is not a root of the polynomial $P(x)$.

Secondly, we observe that if the r_i's are the roots of $P(x)$ (listed with their corresponding multiplicities). So, the problem asks for proving that there exists a **positive** lower bound for the integral

$$\int_{-1}^{1}\prod_i\left|\frac{x-r_i}{r_i}\right|\mathrm{dx}.$$

Our strategy is to show that there exists a subinterval $I \subset [-1,1]$ of length larger than some given positive quantity such that for all points x in I, **each** of the factors $|(x-r_i)/r_i|$ are bounded below by another positive quantity (note that each r_i is nonzero according to our initial assumption as above).

Since $P(x)$ has less than 2020 distinct roots, then there exists an interval $I \subset [0,1/2]$ of length at least $\frac{1}{10^4}$ such that none of the roots of $P(x)$ are within $\frac{1}{10^4}$ of some point contained in I.

Now, for any root r of $P(x)$ and for any point $x \in I$, we claim that
$$\left|\frac{x-r}{r}\right| > \frac{1}{10^4}.$$
Indeed, if $|r| \le 1$, then since $|x-r| > \frac{1}{10^4}$, then indeed $|(x-r)/r| > 1/10^4$. So, assume next that $|r| > 1$; but then
$$\left|\frac{x-r}{r}\right| = \left|1 - \frac{x}{r}\right| \ge 1 - \left|\frac{x}{r}\right| > 1 - \frac{1}{2} > \frac{1}{10^4},$$
as claimed. So,
$$\int_{-1}^{1} \left|\frac{P(x)}{P(0)}\right| \mathrm{dx} \ge \int_I \prod_i \left|\frac{x-r_i}{r_i}\right| > \int_I \left(\frac{1}{10^4}\right)^{2020} \mathrm{dx} = \frac{1}{10^{8084}}.$$

Problem 3. The sequence $\{a_n\}$ satisfies
$$a_1 = 1;\ a_2 = 2;\ a_3 = 24 \text{ and for } n \ge 4:$$
$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$
Prove that for each positive integer n, we have that a_n is an integer multiple of n.

Solution. We let $b_n := a_n/a_{n-1}$ for each $n \ge 2$ and so, for all $n \ge 4$, we have:
$$b_n = 6b_{n-1} - 8b_{n-2}, \text{ where}$$
$$b_2 = 2 \text{ and } b_3 = 12.$$
We solve first for the sequence $\{b_n\}$ whose characteristic roots are 2 and 4 and a simple computation yields that for all $n \ge 2$, we have:
$$b_n = -2^{n-1} + 4^{n-1}.$$
So, using that $a_1 = 1$, we conclude that
$$a_n = \prod_{i=1}^{n-1} (4^i - 2^i).$$
Now, for each positive integer n, we write it as $n = 2^a \cdot b$, where $a \ge 0$ and b is an odd integer. We have that, after denoting by $\phi(m)$ the Euler-totient function corresponding to each integer m,
$$4^{a\cdot\phi(b)} - 2^{a\cdot\phi(b)} \equiv 0 \pmod{n}.$$
Indeed, clearly, the above expression is divisible by 2^a, so we're left to prove that it must also be divisible by b. However,
$$4^{a\phi(b)} - 2^{a\phi(b)} = 2^{a\phi(b)} \cdot \left(2^{a\phi(b)} - 1\right) \equiv 0 \pmod{b},$$
using Euler's theorem because $2^{\phi(b)} \equiv 1 \pmod{b}$ (and then $2^{m\phi(b)} \equiv 1 \pmod{b}$ for any positive integer m). Finally, we observe that
$$a \cdot \phi(b) < n = 2^a \cdot b,$$
because $\phi(b) \le b$ and $a < 2^a$ for any $b \ge 1$ and any $a \ge 0$.

Problem 4. Let $P \in \mathbb{C}[x]$ be a polynomial of degree n such that $P(x) = Q(x) \cdot P''(x)$, where $Q(x)$ is a quadratic polynomial and P'' is the double derivative of

P. Show that if $P(x)$ has at least two distinct roots, then it must have n distinct roots.

Solution. Assume r is a root of $P(x)$ of multiplicity $m \geq 2$. Then $P''(x)$ has the root r with multiplicity $m-2$; therefore, $Q(x)$ must have the root r with multiplicity 2. Furthermore, looking the leading coefficients of both $P(x)$ and of $P''(x)$, we conclude that $Q(x) = \frac{1}{n(n-1)} \cdot (x-r)^2$. Now, we write

$$P(x) = \sum_{i=0}^{n} c_i (x-r)^i;$$

actually, from our assumption, we know that $a_i = 0$ for $0 \leq i < m$ (where $m \geq 2$). Then

$$P''(x) = \sum_{i=m}^{n} i(i-1)c_i (x-r)^{i-2}$$

and then equating $P(x) = \frac{(x-r)^2}{n(n-1)} \cdot P''(x)$ (in their expansions around $x = r$), we get that c_i must be equal to 0 whenever $i < n$, which contradicts the assumption that $P(x)$ has at least two distinct roots. This concludes our proof.