PUTNAM PRACTICE SET 4

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Problem 1. Let $\{F_n\}_{n\geq 1}$ be the Fibonacci sequence, i.e.,

$$F_1 = 1,\ F_2 = 1 \text{ and } F_{n+2} = F_{n+1} + F_n \text{ for each } n \geq 1.$$

Find all positive real numbers a and b with the property that for each $n \geq 1$, we have that $aF_n + bF_{n+1}$ is another element of the Fibonacci sequence.

Problem 2. For any polynomial $P \in \mathbb{C}[x]$ and for each complex number a, we denote by P_a the set of all $z_0 \in \mathbb{C}$ such that $P(z_0) = a$. Let $P, Q \in \mathbb{C}[x]$ such that $P_2 = Q_2$ and $P_5 = Q_5$. Prove that $P = Q$.

Problem 3. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We have a function $f : \mathbb{N}_0^2 \longrightarrow \mathbb{N}_0$ satisfying the following properties:

- $f(0, y) = y + 1$ for each $y \in \mathbb{N}_0$;
- $f(x + 1, 0) = f(x, 1)$ for each $x \in \mathbb{N}_0$; and
- $f(x + 1, y + 1) = f(x, f(x + 1, y))$ for each $x, y \in \mathbb{N}_0$.

Find $f(4, 2019)$.

Problem 4. We consider all possible sequences $\{x_n\}_{n\geq 0}$ of positive real numbers having the properties that $x_0 = 1$ and also that $x_{n+1} \leq x_n$ for each $n \geq 0$.

(I) Prove that for each such sequence $\{x_n\}_{n\geq 0}$, we have that the series

$$\sum_{i=0}^{\infty} \frac{x_i^2}{x_{i+1}}$$

is either divergent to $+\infty$, or it converges to a real number at least equal to 4.

(II) Prove that there exists exactly one such sequence $\{x_n\}_{n\geq 0}$ for which the series

$$\sum_{i=0}^{\infty} \frac{x_i^2}{x_{i+1}}$$

equals 4.