Mathematics 437/537 – Final Examination, December 11, 2006

Instructions

(i) Each question is worth 10 marks.

(ii) Don't rush. Complete solutions to a smaller number of questions will be more valuable than partial solutions to a larger number.

Questions

1. Suppose that x, y, z are positive integers satisfying $(x, y) = 1$ and $xy = z^2$. Show that there are integers u and v for which $x = u^2$ and $y = v^2$.

2. Find all values of a for which the congruence $x^6 \equiv a \pmod{31}$ has integer solutions.

3. Suppose that $p = 8n + 5$ is a prime and that a is a quadratic residue modulo p. Show that the two solutions of $x^2 \equiv a \pmod{p}$ are given by one of the following possibilities: $x \equiv \pm a^{n+1} \pmod{p}$ or $x \equiv \pm 2^{2n+1} a^{n+1} \pmod{p}$.

4. For which of the following values of k does $x^2 + y^2 = k!$ have a solution with x, y integers: (a) $k = 6$, (b) $k = 12$, (c) $k = 24$, (d) $k = 48$?

5. Show that if d is an odd squarefree integer then a necessary condition for the equation $x^2 - dy^2 = -1$ to have an integer solution is that all the primes dividing d must be of the form $4n + 1$.

6. Let $d = 4k^2 + k$, where $k > 1$ is an integer. Show that $x^2 - dy^2 = -1$ has no integer solutions.

7. Show that the only integer solutions of the equation $y^2 + 11 = x^3$ are $(x, y) = (3, \pm 4)$ and $(15, \pm 58)$.

8. Suppose that the positive integer n satisfies $2^n \equiv -1 \pmod{n}$. Let p be the smallest prime divisor of n. Show that $p = 3$.

End of the Examination