Be sure this exam has 3 pages including the cover

The University of British Columbia

Sessional Exams – 2011 Term 1
Mathematics 418 Probability
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This exam consists of **7** questions worth **10** marks each.
No calculators or other aids are permitted.
Show all work and calculations and explain your reasoning thoroughly.
A table of the normal distribution is on the last page.

1. Suppose that a nonempty subset is chosen uniformly at random from a set of n elements, in the sense that every nonempty subset is equally likely to be selected. Let X be the number of elements in the chosen subset. Hint: recall that the mean and variance of a $\mathrm{Bin}(n, p)$ random variable are respectively np and $np(1-p)$.

   (a) Show that
   $$EX = \frac{n}{2 - 2^{1-n}}.$$

   (b) Show that
   $$\mathrm{Var}X = \frac{n2^{2n-2} - n(n+1)2^{n-2}}{(2^n - 1)^2}.$$

2. Suppose that X_1 and X_2 are independent binomial random variables with parameters (n_1, p) and (n_2, p), respectively. Prove that $X_1 + X_2$ is a binomial random variable with parameters $(n_1 + n_2, p)$.

3. A geometric random variable with parameter $p \in (0, 1)$ has probability mass function $P(X = n) = (1-p)^{n-1}p$, $n = 1, 2, 3, \ldots$. Let X_1 and X_2 be independent geometric random variables, both with parameter p. Find:

   (a) $P(X_1 = X_2)$,

   (b) $P(X_1 \geq X_2)$.

4. Let X_n be independent $N(0, 1)$ random variables. Prove that
   $$P\left(\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{2\log n}} = 1\right) = 1.$$
   Hint: Recall that the cumulative distribution function Φ of the standard normal distribution obeys:
   $$(x^{-1} - x^{-3})e^{-x^2/2} < \sqrt{2\pi}[1 - \Phi(x)] < x^{-1}e^{-x^2/2} \quad \text{for } x > 0.$$

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded and is continuous at 0. Suppose also that X_n are random variables such that $X_n \to 0$ in probability (we are *not* assuming almost sure convergence). Prove that $\lim_{n\to\infty} E(f(X_n)) = f(0)$.

6. A certain immigration clerk will answer the telephone only on weekdays at about 10:00 a.m. On any such morning she is equally likely to be at her desk or not. If she is absent no one answers, and days are independent. Assume the line is never busy. If she is at her desk, the time T that she takes to answer her phone is a random variable with distribution function given by $F_T(t) = P(T \leq t) = 0$ for $t \leq 1$ and $F_T(t) = P(T \leq t) = 1 - t^{-1}$ for $t > 1$.

   (a) If you telephone this clerk one morning, and do not hang up, what is the probability that the phone rings for a time R that is at least time s?

   (b) You decide to telephone the clerk one morning, and to hang up at time s if she has not answered by then. If your call is successful, then the expected time for which the telephone rings is $E(R|R < s)$. Show that
   $$E(R|R < s) = \frac{s\log s}{s-1}, \quad \text{for} \quad s > 1.$$
   Hint: the expectation of a nonnegative continuous random variable X is $\int_0^\infty P(X > t)dt$.

7. Ten numbers are rounded to the nearest integer and then summed. Using the central limit theorem, determine the probability that the sum of the rounded numbers will equal the rounded sum of the unrounded numbers. Assume that the roundoffs for the ten numbers are independent and uniformly distributed in $(-0.5, 0.5)$, and hence have variance $\frac{1}{12}$.
For the arithmetic: $\sqrt{0.3} \approx 0.548$.

Table 1: Cumulative distribution function $\Phi(x)$ of standard Normal distribution

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990