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The University of British Columbia

Sessional Exams – 2006 Term 1
Mathematics 418 Probability
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This exam consists of **7** questions worth **10** marks each.
No calculators or other aids are permitted.
Show all work and calculations and explain your reasoning thoroughly.
A table of the normal distribution is on the last page.

1. Prove the identity
$$\sum_{k=0}^{n} \binom{n+k}{k} 2^{-(n+k)} = 1.$$
Hint: Consider a random walk on the square lattice $\mathbb{Z}^2$ which starts at the origin $(0,0)$ and takes independent steps either upward (from (x,y) to $(x,y+1)$) or the the right (from (x,y) to $(x+1,y)$) with equal probabilities $\frac{1}{2}$. Note that it eventually leaves the square $[0,n]\times[0,n]$.

2. Let X and Y be discrete random variables.
   (a) Prove directly from the definition of conditional expectation that
   $$E(XY) = E[XE(Y|X)].$$
   (b) Prove that if
   $$E(Y|X=x) = E(Y) \quad \text{for all } x,$$
   then X and Y are uncorrelated.
   (c) Give a counterexample to show that the converse of (b) is false.

3. Let X be uniform on $[0,1]$ and Y be uniform on $[0,X^2]$.
   (a) Find the marginal density of Y.
   (b) Find $E(X|Y)$.

4. Suppose n darts are thrown at a circular target of radius 1. Assume the darts land at positions which are i.i.d. random variables uniform on the unit disk. Let Z_n be the distance from the centre of the target to the closest dart. Show that $E(Z_n) = \int_0^1 (1-x^2)^n dx$.

5. Consider a branching process whose family sizes have the geometric mass function $f(k) = 2^{-k-1}$, for $k \geq 0$. Let Z_n be the size of the nth generation, and assume that $Z_0 = 1$.
   (a) Show that Z_n has generating function
   $$G_n(s) = \frac{n-(n-1)s}{n+1-ns}.$$
   (b) Let $T = \min\{n : Z_n = 0\}$ be the extinction time. Find $P(T=n)$ for all $n \geq 1$.
   (c) Prove that $P(T < \infty) = 1$.

6. Let $\lambda > 0$. For $n = 1,2,3,\ldots$, let X_n have a Binomial$(n, \lambda/n)$ distribution. Let X have a Poisson(λ) distribution. Prove that X_n converges to X in distribution.

7. An astronomer is interested in measuring, in light years, the distance from her observatory to a distant star. Although she has a measuring technique, she knows that because of changing atmospheric conditions and experimental error, each time a measurement is made it will not yield the exact value but rather an approximate value. As a result the astronomer plans to make a series of measurements and then use the average of these measurements as the estimated value of the actual distance. If the astronomer believes that the values of the measurement errors are not systematic, so that the measurements are described by a random variable with mean d (the true distance) and a variance of 4 light years, use the central limit theorem to determine approximately the number of measurements that should be made to be 95% sure that the estimated distance is accurate to within ± 0.5 light years.

Table 1: Cumulative distribution function $\Phi(x)$ of standard Normal distribution

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990