Be sure that this examination has 3 pages.

The University of British Columbia

Final Examinations - April 2008

Mathematics 401

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Closed book examination Time: $2\frac{1}{2}$ hours

Special Instructions: A two-sided single page of notes is allowed.

Marks

[20] **1.** Consider the following problem for $u(x, y)$ in a square:

$$u_{xx} + u_{yy} + 4u = f(x, y) \qquad \text{in} \quad 0 \leq x \leq \pi\,, \;\; 0 \leq y \leq \pi\,,$$
$$u_x(0, y) = u_x(\pi, y) = 0\,, \qquad u(x, 0) = u(x, \pi) = 0\,.$$

(i) Is there a condition on $f(x, y)$ that is required in order for there to be a solution to this problem? If so, find this condition.

(ii) Show how to represent $u(x, y)$ in terms of either a Green's function or a modified Green's function, which ever is appropriate. (You do not need to calculate this Green's function analytically).

[20] **2.** Consider the following problem for $u(x, y)$ in the unit disk:

$$u_{xx} + u_{yy} - u = f(x, y) \quad \text{in} \quad x^2 + y^2 \leq 1\,; \quad u = h(x, y) \quad \text{on} \quad x^2 + y^2 = 1\,.$$

(i) Show how to represent u in terms of an appropriate Green's function G. Is there a simple analytical formula for G by the method of images?

(ii) In terms of the usual modified Bessel functions, derive the following identity:

$$K_0(R) = \sum_{n=-\infty}^{\infty} e^{in(\phi-\theta)} I_n(r_<) K_n(r_>)\,,$$

where $r_< = \min(r, \rho)$, $r_> = \max(r, \rho)$, and $R = \sqrt{r^2 + \rho^2 - 2r\rho\cos(\theta - \phi)}$. (Hint: you will need the Wronskian relation $I_n'(x)K_n(x) - I_n(x)K_n'(x) = 1/x$.

(iii) By using the identity in (ii), give an infinite series representation for the Green's function that is required in part (i).

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[20] **3.** Let $u = u(x)$ and consider the functional

$$I(u) = \int_0^L F(x, u, u^{'}, u^{''})\, dx\,,$$

over all four times continuously differentiable functions, $u(x)$, satisfying the boundary conditions $u(0) = u(L) = 0$.

(i) Show that the Euler-Lagrange equation associated with $I(u)$ is

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u''}\right) = 0\,.$$

What are the natural boundary conditions for u at $x = 0, L$?

(ii) Suppose that

$$F(x, u, u', u'') = \frac{1}{2}\left[u''\right]^2 + \frac{1}{2}\left[u'\right]^2 - \frac{\sigma}{(1+u)}\,,$$

where σ is a positive constant. Write the associated Euler-Lagrange equation and boundary conditions for u explicitly. (This problem models the deflection of a beam in a micro-electrical-mechanical system).

(ii) Next, consider the eigenvalue problem

$$(p(x)u'')'' = \lambda u\,, \quad 0 \le x \le L; \qquad u(0) = u(L) = u'(0) = u'(L) = 0\,,$$

with $p(x) > 0$ in $0 \le x \le L$. Find a variational principle, together with a simple trial function, that can be used to give an upper bound on the first eigenvalue λ_1 (Do not calculate this bound analytically).

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[20] **4.** Consider the following diffusion equation for $u(x,t)$:

$$u_t = u_{xx} + f(x,t)\,, \qquad 0 \le x < \infty\,, \quad t > 0\,,$$
$$u_x(0,t) = h(t)\,; \qquad u(x,0) = 0\,,$$

with $u \to 0$ as $x \to +\infty$.

(i) Show how to represent $u(x,t)$ in terms of an appropriate Green's function.

(ii) Calculate the Green's function needed in (i) analytically.

[20] **5.** Consider the following eigenvalue problem for $\phi(x,y)$ in the triangular-shaped region $\Omega = \{(x,y)|\ 0 \le x \le 2,\ 0 \le y \le 2,\ y \le 2-x\}$:

$$\phi_{xx} + \phi_{yy} - xy\,\phi + \lambda\phi = 0 \quad \text{in } \Omega\,; \quad \phi = 0 \quad \text{on } \partial\Omega\,.$$

(i) Find explicit upper and lower bounds for the first eigenvalue λ_1 by bounding the coefficient $q(x,y) = xy$ and by bounding the triangular region by circles of appropriate radii. (Hint: You are given that the smallest eigenvalue σ_1 for the Laplacian in a circle of radius one is 5.78).

(ii) Suppose that you had two admissible trial functions $v_1(x,y)$ and $v_2(x,y)$. Explain clearly how you would obtain an upper bound for λ_1 by the Rayleigh-Ritz method.

[100] **Total Marks**

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