The University of British Columbia

Math 401 Final Examination - April 2006

Closed book exam. No notes or calculators allowed.
Answer all 4 questions

1. [25]
(a) For the problem:

$$\begin{aligned} u'' + \pi^2 u &= f(x), \quad 0 < x < 1 \\ u(0) &= 0, \quad u(1) = 0, \end{aligned}$$

determine the solvability condition on $f(x)$.
(b) Assuming this condition is satisfied, calculate the modified Green's function, $G(\xi; x)$.
(c) Find an integral representation of the solution.

2. [25]
(a) Describe how you would use the Green's function method to solve the (3-D) problem,

$$\begin{aligned} L[u] &= \nabla^2 u - k^2 u = f(\underline{x}), \quad -\infty < (x,y) < \infty,\ 0 < z < \infty, \\ u(x,y,z) &= g(x,y), \text{on the plane } z = 0. \end{aligned} \tag{1}$$

and give a representation of the solution in terms of $G(\underline{\xi};\underline{x})$.
b) Find the free-space Green's function, $F(\underline{\xi};\underline{x})$, where $\underline{x} = (x,y,z)$ for the problem (1).

Note that in spherical coordinates, $F(r)$ can be represented as a singular solution of

$$\nabla^2 F - k^2 F = F'' + \frac{2}{r}F' - k^2 F = 0,$$

and a transformation $F(r) = r^{-1}W(r)$ will be useful.

(c) Find the Green's function, $G(\underline{\xi};\underline{x})$ for the problem (1) in terms of $F(r)$.

3. [25]
The region D is the triangle bounded by the lines, $y = \pm x/\sqrt{3}$ and $x = b > 0$, and

$$\begin{aligned} u_{xx} + u_{yy} + 2 &= 0 \text{ in } D, \\ u &= 0 \text{ on } \partial D, \end{aligned} \tag{2}$$

(a) Describe how you would find approximate solutions to (2) using both a Galerkin and a Rayleigh-Ritz method.
(b) Describe how you would find an approximate one-term Kantorovich solution of the form $U(x,y) = (y^2 - x^2/3)V(x)$.

4. [25]

(a) Write down an expression for the Rayleigh Quotient for the general Sturm-Liouville problem:

$$\begin{aligned} (p(x)u')' - q(x)u &= -\lambda r(x)u, \quad 1 \le x \le 2 \\ u(1) &= u(2) = 0 \end{aligned}$$

(b) For $\alpha > 1$, solve

$$\begin{aligned} x^{-1}(x^3 u')' &= -\alpha u, \quad 1 \le x \le 2 \\ u(1) &= u(2) = 0 \end{aligned}$$

You might want to expand it out to help solve it.
Give a rough numerical estimate of α. You can take $\ln 2 \sim 0.7$ and $\pi^2 \sim 10$.
(c) Obtain upper and lower bounds for λ_1 for the eigenvalue problem:

$$\begin{aligned} x^{-1}((x^3+1)u')' &= -\lambda u, \quad 1 \le x \le 2 \\ u(1) &= u(2) = 0 \end{aligned}$$

by using different methods.