Be sure that this examination has 3 pages.

The University of British Columbia

Final Examinations - December 2015

Mathematics 400

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Closed book examination Time: $2\frac{1}{2}$ hours

Special Instructions: Closed Book and Notes. Calculators are not allowed.

Marks

[15] **1.** Consider the following first order PDE for $u = u(x, y)$:

$$u_x + 2xu_y = (3y + x)u \, .$$

(i) Find the general solution to this PDE.

(ii) Find the specific solution to this PDE that satisfies the data

$$u = f(y) \, , \quad \text{on } \; x = 0 \; \text{ for } \; -1 \le y \le 1 \, .$$

In which region of the (x, y) plane is this solution defined?

[20] **2.** Consider axially symmetric diffusion for $u(r, z, t)$ in a finite cylinder of radius $a > 0$ and height $H > 0$ with insulating boundary conditions modeled by

$$u_t = u_{rr} + \frac{1}{r}u_r + u_{zz} \, , \quad 0 \le r \le a \, , \quad 0 \le z \le H \, , \quad t \ge 0 \, ,$$
$$u_z = 0 \quad \text{on } \; z = 0 \text{ and } z = H \, ; \qquad u_r = 0 \quad \text{on } \; r = a \, , \quad u \text{ bounded as } r \to 0 \, ,$$
$$u(r, z, 0) = f(r, z) \, .$$

Determine an eigenfunction expansion representation for the time-dependent solution $u(r, z, t)$, and also calculate the steady-state solution.

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[20] **3.** Assume that $\beta > 0$ and $\alpha \geq 0$ are constants, and consider the following traffic flow model for the density $\rho(x,t)$ of cars given by

$$\rho_t + (2-\rho)\,\rho_x + \alpha\rho = 0\,, \quad -\infty < x < \infty\,, \quad t > 0\,,$$
$$\rho(x,0) = \frac{3\beta^2}{\beta^2 + x^2}\,.$$

(i) **First let $\alpha = 0$**. Determine a parametric form for the solution $\rho(x,t)$. Plot qualitatively the characteristics in the (x,t) plane, and sketch the solution $\rho(x,t)$ versus x at different times. Determine the time t_b as a function of β when the solution first becomes multi-valued.

(ii) **Now let $\alpha > 0$**. Find a value α_c, which depends on β, such that the solution does not become multi-valued for any $t > 0$ if and only if $\alpha > \alpha_c$.

(iii) **Let $\alpha \geq 0$**. Calculate explicitly the total number $N(t)$ of cars on the road, defined by $N(t) = \int_{-\infty}^{\infty} \rho(x,t)\,dx$.

[20] **4.** Consider the diffusion problem for $u(r,\theta,t)$ in a disk of radius a with an inflow/outflow flux boundary condition modeled by

$$u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\,, \quad 0 \leq r \leq a\,, \quad 0 \leq \theta \leq 2\pi\,, \quad t \geq 0\,,$$
$$u_r(a,\theta,t) = f(\theta)\,, \quad u \text{ bounded as } r \to 0\,, \quad u \text{ and } u_\theta \text{ are } 2\pi \text{ periodic in } \theta\,,$$
$$u(r,\theta,0) = g(r,\theta)\,.$$

(i) Write the problem that the **steady-state solution** $U(r,\theta)$ would satisfy. Prove that such a steady-state solution $U(r,\theta)$ does not exist when $\int_0^{2\pi} f(\theta)\,d\theta \neq 0$.

(ii) Assume that $\int_0^{2\pi} f(\theta)\,d\theta = 0$. Calculate an integral representation for the **steady state solution** $U(r,\theta)$ by summing an appropriate eigenfunction expansion.

(ii) Assume that $\int_0^{2\pi} f(\theta)\,d\theta \neq 0$. Determine an approximation to the time-dependent solution $u(r,\theta,t)$ that is valid for large time t.

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[25] **5. Each of these five short-answer questions below is worth 5 points. Very little calculation is needed for any of these problems.**

(i) In the circular disk $0 < r < a$, $0 \leq \theta \leq 2\pi$ find the explicit solution to Laplace's equation for $u(r,\theta)$:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0\,, \quad 0 \leq r \leq a\,, \quad 0 \leq \theta \leq 2\pi\,,$$
$$u(a,\theta) = 2\cos^2(\theta) + 1\,, \quad u \text{ bounded as } r \to 0\,, \quad u \text{ and } u_\theta \;\; 2\pi \text{ periodic in } \theta\,.$$

(Hint: Recall that $\cos^2(\theta) = [1 + \cos(2\theta)]/2$.)

(ii) Outside the sphere $r > a$, where θ is the polar angle with $0 \leq \theta \leq \pi$, find the explicit solution to Laplace's equation $\Delta u = 0$ with $u = 2\cos^2(\theta) + 1$ on $r = a$, $u \to 0$ as $r \to \infty$, and u is bounded at the poles $\theta = 0, \pi$. (Recall that the first three Legendre polynomials are $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = (3x^2 - 1)/2$.)

(iii) Consider the radially symmetric diffusion problem for $u(r,t)$ in the unit sphere $0 < r < 1$, modeled by

$$u_t = D\left(u_{rr} + \frac{2}{r}u_r\right)\,, \quad 0 \leq r \leq 1\,, \quad t > 0$$
$$u(1,t) = 0\,, \quad u \text{ bounded as } r \to 0\,; \qquad u(r,0) = f(r)\,,$$

where $D > 0$ is constant. Show that for long time, i.e. for $t \to +\infty$, that the solution can be approximated by $u(r,t) \approx Ae^{-D\pi^2 t}\sin(\pi r)/r$ for some $A > 0$ to be found.

(iv) Consider the damped wave-equation on a finite interval $0 < x < L$, and with time-periodic forcing modeled by

$$u_{tt} + au_t = c^2 u_{xx} + \sin(\omega t)\,, \quad 0 < x < L\,, \quad t > 0\,,$$
$$u(0,t) = 0\,, \quad u(L,t) = 0\,; \qquad u(x,0) = 0\,, \quad u_t(x,0) = 0\,.$$

Here $c > 0$, while $a > 0$ is small. For what frequencies $\omega > 0$ will we obtain a very large response for u when $a > 0$ is small? (Hint: these are the frequencies where resonance would occur if $a = 0$).

(v) Solve the signalling problem for the wave equation $u(x,t)$ where the signal is applied on a space-time curve as follows:

$$u_{tt} = c^2 u_{xx}\,, \quad c_0 t < x < \infty\,, \quad t > 0\,,$$
$$u(c_0 t, t) = \sin(\omega t)\,, \quad u(x,0) = u_t(x,0) = 0\,.$$

Here $0 < c_0 < c$.

[100] **Total Marks**

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