Be sure that this examination has 3 pages.

The University of British Columbia

Final Examinations - December 2013

Mathematics 400

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Closed book examination Time: $2\frac{1}{2}$ hours

Special Instructions: Closed Book and Notes. Calculators are not allowed.

Marks

[20] **1.** Consider the following first order PDE for $u = u(x, y)$:

$$u_x + 3x^2 u_y = (y + 1)u\,.$$

(i) Find the solution to this PDE if we give the data

$$u = f(y)\,, \quad \text{on } x = 0 \text{ for } -1 \le y \le 1\,.$$

In which region of the (x, y) plane is this solution defined?

(ii) Find a condition on $f(y)$ such that this PDE has a solution which satisfies

$$u = f(y)\,, \quad \text{on } y = x^3 \text{ for } 0 \le x \le 1\,.$$

[20] **2.** Let $c(u)$ be a smooth function satisfying both $c'(u) < 0$ and $c(u) < 0$ for all $u > 0$. Assume also $c(0) = 0$. Consider the nonlinear wave equation for $u(x, t)$ given by

$$u_t + c(u)u_x = 0\,, \quad -\infty < x < \infty\,, \quad t \ge 0\,; \qquad u(x, 0) = e^{-x^2}\,.$$

(i) Sketch a few pictures of u versus x at fixed times to illustrate how u evolves as t increases.

(ii) By using the method of characteristics, derive a formula for the breaking time t_b, defined as the minimum time in $t > 0$ at which the solution $u(x, t)$ becomes multi-valued in x

(iii) Calculate the breaking time t_b explicitly for the special case where $c(u) = -u^4$.

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[20] **3.** Consider Laplace's equation for $u = u(r,\theta)$ in the wedge $0 \le r \le R$, $0 \le \theta \le \alpha$, where (r,θ) are polar coordinates:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0\,, \quad 0 \le r \le R\,, \quad 0 \le \theta \le \alpha\,,$$
$$u(r,0) = 0\,, \quad u(r,\alpha) = 0\,, \quad u(R,\theta) = 1\,, \quad u \text{ bounded as } r \to 0\,.$$

Here $R > 0$ and $0 < \alpha < 2\pi$.

(i) By summing an appropriate eigenfunction expansion, find a compact form for the solution to this PDE.

(ii) From the solution in (i), show that the leading order behavior for u as $r \to 0$ is $u \approx Ar^\beta \sin(\beta\theta)$ for some constants A and β. Calculate the constants A and β explicitly. For what values of the wedge-angle α is the partial derivative u_r bounded as $r \to 0$? (Remark: you can still find β even if you are unable to solve part (i)).

[20] **4.** Consider a sphere of radius a with azimuthal symmetry and with surface potential $u(a,\theta) = f(\theta)$. Then, the solution $u(r,\theta)$ to the axisymmetric Laplace's equation, defined both inside and outside the sphere, satisfies

$$\frac{1}{r^2}\left(r^2 u_r\right)_r + \frac{1}{r^2 \sin\theta}\left(\sin\theta\, u_\theta\right)_\theta = 0\,, \quad 0 < r < \infty\,, \quad 0 < \theta < \pi\,,$$
$$u(a,\theta) = f(\theta)\,, \quad 0 < \theta < \pi\,,$$
$$u \text{ bounded as } r \to 0 \text{ and } r \to \infty\,; \quad u \text{ bounded as } \theta \to 0 \text{ and } \theta \to \pi\,.$$

(i) Imposing that u is continuous across $r = a$ calculate $u(r,\theta)$ for $r > a$ and then for $0 < r < a$.

(ii) In terms of $f(\theta)$, give a formula for the surface charge density $\sigma(\theta)$ defined by

$$\sigma(\theta) = \left.\frac{\partial u}{\partial r}\right|_{r=a^+} - \left.\frac{\partial u}{\partial r}\right|_{r=a^-}\,.$$

(iii) Calculate $\sigma(\theta)$ explicitly when $f(\theta) = \cos(3\theta)$. (Hint: you need the identity $\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos(3\theta)$. The first few Legendre polynomials are given on the next page).

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[20] **5.** Consider axially symmetric diffusion for $u(r,z,t)$ in a finite cylinder with insulating boundary conditions, but with bulk decay, modeled by

$$u_t = u_{rr} + \frac{1}{r}u_r + u_{zz} - \kappa u\,, \quad 0 \le r \le a\,, \quad 0 \le z \le H\,, \quad t \ge 0\,,$$
$$u_z = 0 \quad \text{on } z = 0, H\,, \qquad u_r = 0 \quad \text{on } r = a\,, \quad u \text{ bounded as } r \to 0\,,$$
$$u(r,z,0) = f(r,z)\,.$$

The coefficient of bulk decay, κ, is assumed to be a positive constant.

(i) Determine an eigenfunction expansion representation for the time-dependent solution $u(r,z,t)$.

(ii) From your eigenfunction expansion, find an approximation to the solution that shows how $u \to 0$ as $t \to \infty$.

(iii) Define the mass $M(t)$ by $M(t) \equiv \int_0^H \int_0^a u(r,z,t)\, r\, dr\, dz$. Calculate $M(t)$ for all $t > 0$ in terms of $f(r,z)$ and $\kappa > 0$ (Remark: you do not have to solve (i) to answer this question).

Legendre Polynomial Information: The first few Legendre polynomials $P_n(x)$ are as follows:

$$P_0(x) = 1\,, \qquad P_1(x) = x\,, \qquad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)\,, \qquad P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)\,.$$

[100] **Total Marks**

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