The University of British Columbia

Final Examination - December, 2011

Mathematics 400

Closed book examination **Time: 2.5 hours**

Last Name: ________________ , **First:** ____________ **Signature** ________________

Student Number ________________

Special Instructions:

No books, notes or calculators are allowed.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.

  (b) Speaking or communicating with other candidates.

  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		20
2		20
3		20
4		20
5		20
Total		100

1.[20] Consider the equation for $u(x, y)$:

$$xu_x - yu_y = u^2.$$

(i) Determine the characteristics and the general solution.
(ii) Find the explicit solution satisfying $u = 1$ on $x = y^2$.
(iii) For the conditions given in (ii), plot the initial curve, the characteristic curves and the curve along which u becomes infinite.

2. [20]

(i) Find the (implicit) solution to the equation:

$$u_t + u^3 u_x = 0, \text{ with } u(x,0) = f(x) \text{ for } -\infty < x < \infty.$$

(ii) Find the condition for a shock to form when $f(x) = e^{-x^2}$

(iii) Find the time when a shock first forms when $f(x) = e^{-x^2}$

(iii) If the initial condition is changed to

$$f(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

find and sketch the shock trajectory and characteristics.

3. [20]

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0.$$

(i) Classify the equation and put in canonical form.
(ii) Find the general solution.

$$\begin{aligned}
0 &= Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F \\
0 &= au_{\xi\xi} + 2bu_{\xi\eta} + cu_{\eta\eta} + du_\xi + eu_\eta + f \\
a &= A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2 \\
b &= A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + C\eta_y\xi_y \\
c &= A\eta_x^2 + 2B\eta_x\eta_y + C\eta_y^2 \\
d &= A\xi_{xx} + 2B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
e &= A\eta_{xx} + 2B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y
\end{aligned}$$

4. **[20]** Consider the eigenvalue problem:

$$\begin{aligned} x\phi'' + \phi' + \frac{\lambda}{x}\phi &= 0, \ 1 < x < e, \\ \phi(1) &= 0, \ \phi'(e) = 0. \end{aligned}$$

(i) Write the problem in Sturm-Liouville form and state the orthogonality condition.
(ii) Prove (without solving the equation explicitly) that the eigenvalues satisfy $\lambda > 0$.
(iii) Determine the eigenvalues and eigenfunctions explicitly.

5. **[20]** Suppose $u(x, y)$ satisfies

$$\begin{aligned} u_{xx} + u_{yy} &= 0,\ 0 < y < \pi,\ x > 0, \\ u(x, 0) &= u(x, \pi) = 0 \text{ and } u(0, y) = h(y) \\ & u \text{ bounded as } x \to \infty. \end{aligned}$$

(i) Find $u(x, y)$ using separation of variables for general $h(y)$ and write the solution in the form

$$u(x, y) = \int_0^\pi G(x, y, s) h(s) ds.$$

(ii) Find $u(x, y)$ in as explicit a form as you can (i.e. sum the series) for general $h(y)$. Note that

$$\begin{aligned} \sin(A + B) &= \sin A \cos B + \cos A \sin B \\ \cos(A + B) &= \cos A \cos B - \sin A \sin B. \end{aligned}$$