The University of British Columbia
Final Examination - December, 2010
Mathematics 400

1. For the equation

$$xu_x + yu_y = xe^{-u}.$$

(i) Find a parametric solution.
(ii) Determine the characteristics and the find the general solution.
(iii) Find the explicit solution with initial data $u = 0$ on $y = x^2$ for $x > 0$.
(iv) For the conditions given in (iii), plot the initial curve, the characteristic curves and the curve along which u becomes infinite for $x > 0$.

2.
(i) Find the pre-shock solution to the equation:

$$u_t - u^4 u_x = 0, \text{ with } u(x,0) = f(x) \text{ for } -\infty < x < \infty.$$

(ii) For the initial condition

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 < x < 1 \\ 0 & x > 1 \end{cases}$$

find the solution in each region of the (x,t) plane and sketch the shock trajectory and characteristics before the shock and fan intersect.
(iii) At what (x,t) do the shock and fan intersect?

3.

$$u_{xx} + \frac{10}{3}u_{xy} + u_{yy} + \sin(x+y) = 0.$$

(i) Classify the equation and put in canonical form.
(ii) Find the general solution.

4. Consider the eigenvalue problem:

$$\begin{aligned} (x+2)u'' - u' + \frac{\lambda}{x+2}u &= 0, \ 0 < x < 1, \\ u(0) &= u(1) = 0. \end{aligned}$$

(i) Write the problem in Sturm-Liouville form and state the orthogonality condition.
(ii) Prove (without solving the equation explicitly) that the eigenvalues satisfy $\lambda > 0$.
(iii) Determine the eigenvalues and eigenfunctions explicitly when $\lambda > 1$.

5. Consider the following problem *exterior* to a circle of radius R.

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \text{ in } r \geq R, \; 0 \leq \theta \leq 2\pi, \\ u(R,\theta) &= f(\theta), \; u \text{ bounded as } r \to \infty, \\ & \quad u, \; u_\theta, \; 2\pi \text{ periodic in } \theta. \end{aligned}$$

(i) Derive a solution for $u(r,\theta)$ by separation of variables.
(ii) Find $u(r,\theta)$ in as explicit a form as you can (i.e. sum the series) for general $f(\theta)$.
(iii) Calculate $\lim_{r\to\infty} u(r,\theta)$.