The University of British Columbia
Final Examination - December 9, 2005

Mathematics 400

Instructor: Dr. A. Cheviakov

Closed book examination Time: 2.5 hours

Name ______________________ **Signature** ______________________

Student Number ______________________ **Section** 101 102 (circle one)

Special Instructions:

- Be sure that this examination has 7 pages. Write your name on top of each page.

- Submit only this booklet, with solution written in space provided (you may use adjacent page(s)). Clearly outline answers. Solutions on scratch paper will not be graded.

- A 2-sided self-prepared Letter-size formula sheet is allowed. No calculators or notes are permitted.

- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

- Each candidate should be prepared to produce her/his library/AMS card upon request.
- No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of examination.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Making use of any books, papers, or memoranda, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates.
- Smoking is not permitted during examinations.

1		15
2		20
3		20
4		25
5		20
Total		100

[15 pts] Problem 1.
Find the solution $y(x,t)$ of the nonlinear PDE problem

$$y_t + (1 - 3y^2)y_x = 0 \quad -\infty < x < +\infty, \quad t > 0$$

$$y(x,0) = \begin{cases} \sqrt{2/3}, & x < 0; \\ 0, & x > 0. \end{cases}$$

Provide a plot showing characteristics.

[20 pts] Problem 2. Consider a 2-dimensional problem in polar coordinates:

$$u_{tt}(r,\phi,t) = c^2 \Delta u(r,\phi,t), \quad 0 < r < a, \quad 0 \le \phi \le \pi/6, \quad t > 0; \tag{1}$$

$$u(a,\phi,t) = u(r,0,t) = u(r,\pi/6,t) = 0,$$

$$u(r,\phi,0) = f(r,\phi), \quad u_t(r,\phi,0) = 0.$$

Here $a > 0, c > 0$ are constants.
(i) Draw the domain. Solve using separation of variables, to determine the membrane position $u(r,\phi,t)$ for all $t > 0$.
(ii) Write down natural frequencies of oscillations.
(iii) The above equation (1) is a particular case ($A(r,\phi) = c^2, B(r,\phi) = 1$) of a more general linear equation

$$u_{tt}(r,\phi,t) = \frac{1}{B(r,\phi)} \operatorname{div}[A(r,\phi) \operatorname{grad} u(r,\phi,t)]. \tag{2}$$

Separate variables in this equation. Write the eigenvalue problem for the spatial part (use same boundary conditions as for (1) above.)

- Under what conditions on $A(r,\phi), B(r,\phi)$ does this eigenvalue problem have a basis of eigenfunctions?
- State the orthogonality condition for the eigenfunctions.

[20 pts] Problem 3. Consider heat conduction problem in a 2D infinite strip of width H.

$$u_t = k(u_{xx} + u_{yy}), \quad 0 < y < H, \quad -\infty < x < +\infty, \quad t > 0$$

$$u_y(x, 0, t) = u_y(x, H, t) = 0, \qquad u(x, y, 0) = f(x, y).$$

Here $u(x, y, t)$ is temperature, and $f(x, y)$ a smooth function absolutely integrable in the strip.
(i) Solve the problem to find $u(x, y, t)$ for $t > 0$.
(ii) Find the equilibrium heat distribution as $t \to \infty$.
(iii) Find the exact form of the solution, if $f(x, y) = e^{-x^2} \sin(2\pi y/H)$.

[25 pts] Problem 4. Consider the Linear Telegraph Equation problem:

$$u_{xx}(x,t) = \alpha u_{tt}(x,t) + \beta u_t(x,t) + \gamma u(x,t), \quad x > 0,\ t > 0$$

$$u(0,t) = f(t), \quad u(x,0) = u_t(x,0) = 0.$$

Here $\alpha, \beta, \gamma \geq 0$ are constants.
This PDE problem describes electromagnetic signal transmission along a semi-infinite cable; x is a coordinate along the cable, and $t \geq 0$ time. $u(x,t)$ is bounded for all $x, t > 0$.

(i) Find the Laplace transform (in time) $U(x,s)$ of the solution $u(x,t)$.
(ii) Suppose $\beta^2 = 4\alpha\gamma$. Find the solution $u(x,t)$. Write it in the form that explicitly contains necessary Heaviside function(s).
(iii) Verify by substitution that your solution satisfies the initial-boundary value problem.
(iv) Specify another set of constants α, β, γ ($\beta^2 \neq 4\alpha\gamma$), for which the solution $u(x,t)$ can be explicitly found. Find it.

[20 pts] Problem 5. The electric potential $u(x, y)$ in the domain $\mathcal{D}$ between two semi-infinite dielectric plates (see the picture) satisfies the Laplace equation $u_{xx} + u_{yy} = 0$. The potential on the plate P_1 is given by $u_1(r) = e^{-r}$, the potential on the plate P_2 is given by $u_2(r) = e^{-r^2}\sin(3r)$. ($r$ is distance from the origin along each plate.)
Determine $u(x, y)$ inside $\mathcal{D}$. Final answer may contain integrals.