Full Name: ______________________ Signature: ______________________

Student Number: ______________________

Math 361 Final Exam April 2009 2.5 hours.

Instructions: There are **7 pages** in this test including this cover page.
Ensure that **your *full* name and student number** appear on this page.

- No calculators, books, notes, or electronic devices of any kind are permitted.
- Messy work will not be graded. **Read each question carefully** to be sure you are answering the question being asked.

Rules governing formal examinations:

1. Every student must be prepared to produce, upon request, a UBC ID card;
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions;
3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination;
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action;
   (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
   (b) Speaking or communicating with other candidates;
   (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received;
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator; and
6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Part A	Max	Actual
1	3	
2	3	
3	3	
4	3	
Total	12	

Part B	Max	Actual
1	10	
2	15	
3	25	
Total	50	

A. Short-answer questions

For multilple choice questions, circle only one option. For fill-in-the-blank questions, fill in the blank(s).

1. For the system $x' = -x^2 - y$, $y' = \alpha x + \beta y$, the steady state at the origin is guaranteed to be a stable spiral provided

   (a) $\alpha > \beta^2/4$ and $\beta < 0$. (b) $\beta < \alpha^2/4$ and $\beta < 0$. (c) $\alpha < \beta^2/4$ and $\beta < 0$.

   (d) $\alpha < 4\beta^2$ and $\beta > 0$. (e) $\alpha < \beta^2/4$ and $\beta > 0$.

2. The maximum number of inflection points that a solution to the equation

$$\frac{dx}{dt} = -x^4 + 2x^2 + 1$$

   can have is ______. These inflection points occur at the following values of x: ________________.

3. An enzyme (E) with two binding sites for a substrate (S) turns that substrate into a product according to the following reaction scheme:

$$\begin{array}{ccccc} 2S + E & \rightleftharpoons & S + C_1 & \rightleftharpoons & C_2 \\ & & \downarrow & & \downarrow \\ & & S + P + E & & P + C_1 \end{array}$$

   For such a reaction scheme, provided the total concentration of enzyme is small compared to the total concentration of substrate, the formation of product is best described by which of the following options? Assume that the parameters (v, w, K, L) are defined appropriately in terms of the (unspecified) rate constants.

   (a) $\dfrac{dP}{dt} = vS$. (b) $\dfrac{dP}{dt} = \dfrac{vS}{K + S}$. (c) $\dfrac{dP}{dt} = \dfrac{vS^2}{K + S^2}$. (d) $\dfrac{dP}{dt} = \dfrac{vS + wS^2}{K + LS + S^2}$.

4. Consider the differential equation

$$\frac{dS}{dt} = b - \frac{vS + wS^2}{K + LS + S^2}.$$

   List three legitimate and independent natural scales for S: ______, ______, ______.

   List three legitimate and independent natural scales for t: ______, ______, ______.

B. Long-answer questions

1. Consider the Ricker map define by

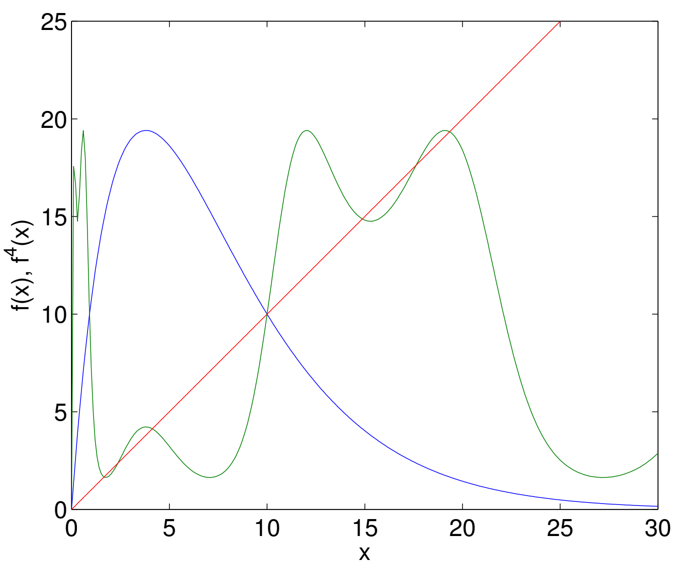
$$x_{n+1} = f(x_n) = e^{r\left(1-\frac{x_n}{K}\right)} x_n.$$

The figure below shows $f(x)$ and $f^4(x)$.

(a) (4 pts) For the parameter values used in the figure, does the Ricker map have a 2-cycle? If so, is it stable or unstable? Does it have a 4-cycle? If so, is it stable or unstable?

(b) (4 pts) Choose one cycle and draw the cobweb on the graph of $f(x)$. Include arrow to indicate the direction of cobwebbing. Circle the fixed point.

(c) (2 pts) What value of K was used for this plot? How do you know?



2. The *lac* operon is a bacterial genetic regulatory module used to control the production of a lactose-digesting enzyme. When the extra-cellular concentration of lactose is low, the production of enzyme shuts off but when external lactose is high, production turns on. The following nondimensionalized system of equations has been proposed as a model for this switch.

$$\begin{aligned}\frac{dx}{d\tau} &= -x + \frac{1}{A}y \\ \frac{dy}{d\tau} &= B + \frac{x^2}{1+x^2} - Cy.\end{aligned}$$

where x is the nondimensional intracellular concentration of lactose and y is the nondimensional concentration of an enzyme responsible for the transport of lactose into the cell.

(a) (10 pts) Sketch the phase plane. Adopt appropriate parameter values of B and C (no need to specify values, just draw the phase plane appropriately) so as to make the system a switch. Use a value of A that puts the system in the "middle" of the switch. Determine stability of fixed points graphically and denote any stable steady state as a filled dot and unstable as empty. Include some points on the axes for scale.

(b) (5 pts) For fixed values of B and C in the "switch" regime, sketch the bifurcation diagram for the parameter A, using the y component of the steady state for the vertical axis. Include some points on the axes for scale.

(c) (bonus) Derive a condition on B and C that ensures the system acts as a switch. I suggest you leave this problem until you are done with everything else as it is a bit technical.

3. The following system of equations is intended to model a whale-hunting population of humans (H) with a limited number of sea-going vessels and the whale population (G).

$$\frac{dH}{dt} = -aH + bG\frac{H^4}{K^4 + H^4}$$
$$\frac{dG}{dt} = cG - dG\frac{H^4}{K^4 + H^4}$$

where all parameters are positive. Assume that $\frac{3}{4} < \frac{c}{d} < 1$ and $d \ll a$.

(a) (5 pts) Explain what each of the four terms in the equations represents.

(b) (5 pts) Define variables h, g and τ and parameters ϵ and α to get the nondimensional version of the equations

$$\frac{dh}{d\tau} = -h + g\frac{h^4}{1 + h^4}$$
$$\frac{dg}{d\tau} = \epsilon\left(\alpha g - g\frac{h^4}{1 + h^4}\right).$$

(c) (10 pts) Draw the phase plane using the techniques of multiple time scales. Include steady states, fast and slow manifold (with direction vectors on both).

(d) (3 pts) Classify all steady states based on the multiple-time-scale-phase plane (no need to linearize, just use the fast-slow analysis).

(e) (2 pts) In biological terms, explain what eventually happens to all solutions if $\alpha = c/d < 3/4$?